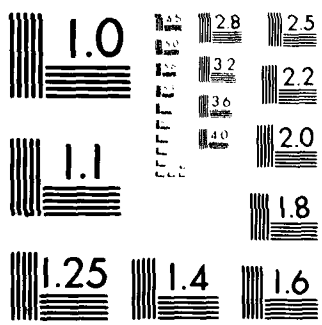
AD-A086 974 PENNSYLVANIA STATE UNIV UNIVERSITY PARK APPLIED RESE--ETC F/G 12/1
WAVES, CAUSTICS AND THOM'S THEOREM.(U)
MAR 80 A D GORMAN N00024-79-C-6043
UNCLASSIFIED ARL/PSU/TM-80-26 NL

1 OF 1
AD
A086974

END
DATE
FILMED
9-80
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

LEVEL II

12

ADA086974

6 WAVES, CAUSTICS AND THOM'S THEOREM.

9 Doctoral thesis,

10 Arthur Daniel Gorman

14 ARL/PSU/TM-80-26

Technical Memorandum
File No. TM 80-26
February 28, 1980
Contract No. N00024-79-C-6043

15

Copy No. 5

11 5 Mar 80

12 81

The Pennsylvania State University
Institute for Science and Engineering
APPLIED RESEARCH LABORATORY
Post Office Box 30
State College, PA 16801

NAVY DEPARTMENT

NAVAL SEA SYSTEMS COMMAND

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER TM 80-26	2. GOVT ACCESSION NO. AD-A086 974	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) WAVES, CAUSTICS AND THOM'S THEOREM		5. TYPE OF REPORT & PERIOD COVERED PhD Thesis, May 1980
		6. PERFORMING ORG. REPORT NUMBER TM 80-26
7. AUTHOR(s) Arthur Daniel Gorman		8. CONTRACT OR GRANT NUMBER(s) N00024-79-C-6043
9. PERFORMING ORGANIZATION NAME AND ADDRESS The Pennsylvania State University Applied Research Laboratory P. O. Box 30, State College, PA 16801		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Sea Systems Command Department of the Navy Washington, DC 20362		12. REPORT DATE March 5, 1980
		13. NUMBER OF PAGES 89 pages & figures
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified, Unlimited
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited, per NSSC (Naval Sea Systems Command), 4/14/80

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

waves, helmholtz, inhomogeneous, asymptotic, series

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The technique of Maslov for determining the first term in the asymptotic series solution of linear partial differential equation is extended to determine the full asymptotic series solution of the reduced Helmholtz equation for scalar waves propagated in an inhomogeneous medium. The algorithm requires the characterization of the medium by a canonical phase function determined from a criterion derived from Thom's classification of singularities.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ABSTRACT

The technique of Maslov for determining the first term in the asymptotic series solution of linear partial differential equations is extended to determine the full asymptotic series solution of the reduced Helmholtz equation for scalar waves propagated in an inhomogeneous medium. The algorithm requires the characterization of the medium by a canonical phase function determined from a criterion derived from Thom's classification of singularities.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

∇	del operator
∇^2	Laplacian
$\sim$	asymptotic equality
$O(\tau^{-\infty})$	asymptotic solution, i.e., a solution in the limit as τ becomes infinite
f^{-1}	the inverse function of f (unless otherwise noted)
∂_i	the first partial derivative in the i-th direction
$\partial_{ij}{}^2$	the first partial derivative in each of the i-th and j-th direction
$\partial_i{}^k$	the k-th partial derivative in the i-th direction
D^k	k applications of the differential operator D
$< >$	the inner product
$\dfrac{\partial(\alpha_1,\alpha_2)}{\partial(\beta_1,\beta_2)}$	the Jacobian
$d^2\phi$	the Hessian (here, of ϕ)
$\Gamma(k)$	the Gamma Function
sgn	the signum operator, i.e., the number of positive eigenvalues less the number of negative eigenvalues of the matrix to which it is applied
$\hat{1}$	the identity operator

Special symbols and notation:

a dot appearing above a quantity indicates the time derivative of the quanity

a Roman numeral appearing above a function, e.g., $f(t)^{IV}$, indicates the appropriate derivative with respect to the argument

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to his committee: Professors Gordon N. Fleming, Milton Cole and Peter B. Shaw of the Department of Physics and, most especially, Robert Wells of the Department of Mathematics. Their advice and encouragement were noteworthy throughout this research. In addition, the author wishes to acknowledge the support of the Applied Research Laboratory of the Pennsylvania State University under contract of the Naval Sea Systems Command.

CHAPTER I

BACKGROUND AND RESEARCH OBJECTIVE

1.1 Introduction

Both partial differential equations and topology are closely related to classical mechanics, the former through Hamilton-Jacobi theory and the latter through global analysis. In Hamilton-Jacobi theory, a configuration space differential equation is given a phase space representation. In global analysis, solutions of differential equations are studied geometrically in phase space. In this investigation, the asymptotic series solution of a particular partial differential equation, the reduced Helmholtz wave equation, is determined using topological considerations in an essential way.

1.2 Background

Wave propagation at a definite frequency, ω, is commonly represented by a partial differential equation of the form

$$\nabla^2\psi(\bar{r},t) = \frac{f(\bar{r},\omega)}{c^2} \frac{\partial^2\psi}{\partial t^2} , \tag{1.2.1}$$

where $\psi(\bar{r},t)$ is the wave function, $\bar{r}$ refers to the spatial coordinates, t is the time, $f(\bar{r},\omega)$ is called the profile characterizing the inhomogeneity of the medium and c is the phase velocity when the medium is homogeneous, i.e., $f(\bar{r},\omega) \equiv 1$. If the medium is non-dispersive, the profile is frequency independent, i.e., $f(\bar{r},\omega) = f(\bar{r})$. For waves transmitted at a particular frequency, the time dependence may be represented by $\exp\{i\omega t\}$. If, for a non-dispersive medium,

$$\psi(\bar{r},t) = \psi(\bar{r})\exp\{i\omega t\} ,$$

Equation (1.2.1) becomes

$$\nabla^2\psi(\bar{r}) + \tau f(\bar{r})\psi(\bar{r}) = 0 \ , \tag{1.2.2}$$

where

$$\tau = \omega^2/c^2 \ . \tag{1.2.3}$$

This is the reduced Helmholtz wave equation, a second order partial differential equation with a variable coefficient. Although no general technique exists for solving such equations [42], various techniques do exist for constructing solutions valid under specific assumptions [11, 46]. One such technique, valid at high frequencies, is the approximation of $\psi(\bar{r})$ by an asymptotic series.

In the asymptotic series solution of Equation (1.2.2), it is assumed that

$$\psi(\bar{r}) \sim \exp\{i\tau\phi(\bar{r})\} \sum_{k=0} a_k(\bar{r})\tau^{-k} \ , \tag{1.2.4}$$

as $\tau \to \infty$. Substitution of Equation (1.2.4) into Equation (1.2.2), regrouping by powers of τ, then equating the coefficients of the powers of τ to zero yields

$$|(\nabla\phi)|^2 - f(\bar{r}) = 0 \tag{1.2.5}$$

and

$$2(\nabla\phi) \cdot (\nabla a_k) + (\nabla^2\phi)a_k = -\nabla^2 a_{k-1}; \ k = 0,1,2,\ldots; \ a_{-1} = 0. \tag{1.2.6}$$

Equation (1.2.5) is the eikonal equation; Equation (1.2.6) is the transport equation. $\psi(\bar{r})$ is obtained by solving the eikonal equation for $\phi(\bar{r})$, then substituting $\phi(\bar{r})$ into the transport equation determines the $a_k(\bar{r})$ recursively. The first term in the series, $a_o(\bar{r})\exp\{i\tau\phi(\bar{r})\}$, is the JWKB approximation [30]. The remaining terms may be regarded as higher order corrections to JWKB approximation [52].

Two basic difficulties with this approach are noteworthy. First, $\phi(\bar{r})$ must be determined from a first order nonlinear partial differential equation. As no general technique exists for solving such equations, $\phi(\bar{r})$ itself may require an approximate solution. The second difficulty involves singular, or turning, points and is most clearly illustrated by considering the eikonal and transport equations in one dimension, i.e.,

$$\left(\frac{d\phi}{dx}\right)^2 - f(x) = 0 \tag{1.2.7}$$

and

$$2\frac{d\phi}{dx}\ \frac{da_k}{dx} + \frac{d^2\phi}{dx^2}\, a_k = -\frac{d^2 a_{k-1}}{dx^2}\ ;\ k = 0,1,2,\ldots;\ a_{-1} = 0. \tag{1.2.8}$$

Equation (1.2.7) can be integrated to obtain

$$\phi(x) = \pm \int^x f(s)^{1/2} ds\ .$$

Substituting for $\phi(x)$ in Equation (1.2.8) determines

$$a_o(x) = f(x)^{-1/2}\ .$$

If at $x = x_o$, $\dfrac{d\phi(x_o)}{dx} = 0$, then $f(x_o) = 0$ and $a_o(x_o)$ is unbounded. Points such as x_o are called singular points. Points, x_i, at which $a_o(x)$ does not become unbounded are called regular points.

More physically, in two or three dimensions, a finite source produces a field of waves which can be described at high frequencies by a family of trajectories. The contact of neighboring trajectories near turning points leads to a focusing which effects a theoretically infinite field intensity on an envelope defined by the locus of turning (singular) points. Beyond this boundary, in theory, no energy is

propagated. Heuristically, a caustic is the higher dimensional analog of a turning point; analogously, the above asymptotic expansion is not valid on caustics [12].

The asymptotic expansion technique has been modified by several investigators to extend its validity to include caustic regions. One approach is to regard the caustic as a boundary on which a separate asymptotic expansion is determined; however, this boundary expansion must be matched termwise with expansions valid in non-caustic regions [46]. Another approach is to determine a uniformly valid asymptotic series, i.e., an asymptotic series which satisfies the reduced Helmholtz wave equation to arbitrarily high order in τ^{-1} in regions which contain caustics. The procedure involves transforming $a(\bar{r})$ to an appropriate special function, e.g., Bessel, Hankel, etc., and determining the asymptotic series in terms of the special function. The technique applies only to caustics of simple geometry [54].

An alternate formulation is an integral representation for $\psi(\bar{r})$, i.e.,

$$\psi(\bar{r}) = \iiint A(\bar{r},\bar{p},\tau)\exp\{i\tau\phi(\bar{r},\bar{p})\}d\bar{p} \tag{1.2.9}$$

where

$$A(\bar{r},\bar{p},\tau) \sim \sum_{k=0} A_k(\bar{r},\bar{p})\tau^{-k} .$$

In such a representation $A(\bar{r},\bar{p},\tau)$ may be regarded as an amplitude and $\phi(\bar{r},\bar{p})$ as a phase; $\bar{p}$ is the wave vector, i.e., the normal to the wavefronts, the surfaces of constant phase [14, 33]. More physically, the field at any point is represented as a superposition of incident plane

waves summed over all directions. Consequently, Equation (1.2.2) has an asymptotic solution of the form

$$\psi(\bar{r}) - \iiint A(\bar{r},\bar{p},\tau)\exp\{i\tau\phi(\bar{r},\bar{p})\}d\bar{p} = O(\tau^{-\infty}) \tag{1.2.10}$$

as $\tau \to \infty$. The full asymptotic series of $\psi(\bar{r})$ is then the sum of the asymptotic series of each integral

$$\tau^{-k} \iiint A_k(\bar{r},\bar{p})\exp\{i\tau\phi(\bar{r},\bar{p})\}d\bar{p} \ .$$

Maslov (as explained by Arnold [10], Eckmann and Seneor [24] and Guillemin and Sternberg [28]) extended the significance of this representation. He noted that if the wave vector, $\bar{p}$, was regarded as a momentum, the phase, $\phi(r,p)$, could be determined from considerations in Hamilton-Jacobi theory, rather than directly from the eikonal equation. Maslov also determined a formula for the first term in the asymptotic series, i.e., the asymptotic limit, at regular points. The approach of Maslov was generalized to include singular points by Arnold [2-9].

The entire asymptotic series at regular points of the integral in Equation (1.2.9) may be determined by using the stationary phase technique [11]. There are two basic approaches to the classical technique. One involves transforming the integral to the form

$$\int \exp\{i\tau u\}dg(u) \ ,$$

where $g(u)$ is the integral of $A(\bar{r},\bar{p},\tau)$ over the region $\phi(\bar{r},\bar{p}) \leq u$ [32]. The asymptotic behavior is then determined by studying $g(u)$. The other approach is to investigate the asymptotic behavior by studying

the integral in Equation (1.2.9) directly [15]. In this research, the latter approach is followed.

The algebraic complexity of applying the stationary phase technique to multiple integrals has been noted [19]. Specifically, explicit coordinate transformations which carry the integrals to tractable forms must be determined. Consequently, the technique is most clearly illustrated by considering the simple integral

$$I = \int_{-\infty}^{\infty} A(x)\exp\{i\tau\phi(x)\}dx \ , \tag{1.2.11}$$

where A(x) and all its derivatives are bounded throughout the region of integration, A(x) is analytic near the regular stationary points of $\phi(x)$, *i.e.*, x_o such that $\frac{d\phi(x_o)}{dx} = 0$, $\frac{d^2\phi(x_o)}{dx^2} \neq 0$, and $\phi(x)$ has no poles in the region of integration. The basic principle underlying the technique is that as $\tau \to \infty$, the principal contribution to the integral comes from the neighborhood of the stationary point [15]. The technique begins by transforming the integral in Equation (1.2.11) to the form

$$\tilde{I} = \exp\{i\tau\phi(x_o)\}\int_{-\infty}^{\infty} \tilde{A}(u)\exp\{i\tau u^2\}du \ , \tag{1.2.12}$$

where $\tilde{A}(u) = A(x(u))dx/du$. $\tilde{A}(u)$ is then Taylor expanded so that

$$\int_{-\infty}^{\infty} \tilde{A}(u)\exp\{i\tau u^2\}du = \tilde{A}_o\int_{-\infty}^{\infty} \exp\{i\tau u^2\}du + \frac{1}{2}\int_{-\infty}^{\infty} u\bar{A}(u)\exp\{i\tau u^2\}du, \tag{1.2.13}$$

where $\tilde{A}_o = a(x_o)\{2/\frac{d^2\phi(x_o)}{dx^2}\}^{1/2}$ and $\bar{A}(u)$ is the remainder of the

Taylor series less a factor of $\frac{1}{2}u$. The first integral on the right-hand side of Equation (1.2.13) is determined by complex integration (Appendix B). A partial integration of

$$\frac{1}{2}\int_{-\infty}^{\infty} u\bar{A}(u)\exp\{i\tau u^2\}du$$

yields

$$\frac{1}{2}\int_{-\infty}^{\infty} u\bar{A}(u)\exp\{i\tau u^2\}du = (i\tau)^{-1}\exp\{i\tau u^2\}\bar{A}(u)\Big|_{-\infty}^{\infty} - (i\tau)^{-1}\int_{-\infty}^{\infty}\exp\{i\tau u^2\}\frac{d\bar{A}(u)}{du}du =$$

$$-(i\tau)^{-1}\int_{-\infty}^{\infty}\exp\{i\tau u^2\}\frac{d\bar{A}(u)}{du}\,du\ ,$$

where the last equality follows from the boundedness of A(x) and its derivatives [28]. Additional terms in the series are determined by repeating the procedure with $\frac{d\bar{A}(u)}{du}$ as the amplitude. However, when $\frac{d^2\phi(x_o)}{dx^2} = 0$, $\tilde{A}_o$ is unbounded; hence, the technique must be modified to remain valid. Analogous to the above, those stationary points at which the technique must be modified to remain valid are called singular points. Those stationary points at which the classical technique is valid are called regular points.

A major generalization of the technique is due to Ludwig [34]. He determined a uniformly valid asymptotic series for regions containing simple caustic geometries by transforming $\phi(\bar{r},\bar{p})$ to an equivalent canonical form. The series is then expressed as the sum of the transformed integrals. Ludwig applied this approach near caustics of simple

geometry by transforming $\phi(\bar{r},\bar{p})$ to an Airy function. The series is then the sum of the Airy integrals, which have been tabulated [1]. In his formulation, however, Ludwig obtained $\phi(\bar{r},\bar{p})$ from the eikonal equation in momentum space.

Duistermaat [22,23], extending both the work of Maslov and Ludwig, developed a formulation that superseded all previous approaches. Maslov and Arnold [3-10] realized that, at singular stationary points, $\phi(\bar{r},\bar{p})$ was topologically equivalent to a canonical Thom potential. The equivalence is determined by the geometry of the caustic. (The Airy function is the lowest order Thom potential.) Duistermaat, however, realized that the transformation to Thom potentials led to uniformly valid asymptotic series for regions containing any physically realizable caustic geometry, thus extending Ludwig's formulation. (Duistermaat also extended Maslov's formula for the asymptotic limit to determine the full asymptotic series at regular stationary points.) As in Ludwig's approach, this asymptotic series is determined as a sum of the transformed integrals. Unlike Ludwig's approach, however, Duistermaat's algorithm was not entirely explicit. Neither were the coordinate transformations carrying $\phi(\bar{r},\bar{p})$ to the Thom potentials explicitly given, nor were the final canonical integrals evaluated.

1.3 Scope of This Investigation

In this investigation, phase space analysis is used to determine explicitly the asymptotic solution of the reduced Helmholtz wave equation. Each field point is represented by the range, i.e., the distance between two points on a horizontal plane, and the depth. Hence,

two coordinates suffice to specify the spatial variation of $\psi(\overline{r})$. For computational purposes, a common technique is not to represent the medium as a continuum but as a series of horizontal layers [17]. Rather than characterizing the medium with a single, depth-dependent profile, each layer is characterized separately by a linear approximation to the profile valid only within the layer. In this investigation, however, the medium is characterized by a single, depth-dependent profile, $f(x)$, which is assumed to be invertible, i.e., $x = f^{-1}(y)$ is defined, except at a finite number of isolated points. (At such points, the algorithm detailed in Chapter VII does not apply.) Thus, the reduced Helmholtz equation considered here is

$$\frac{\partial^2\psi}{\partial x^2}(x,y) + \frac{\partial^2\psi}{\partial y^2}(x,y) + \tau f(x)\psi(x,y) = 0 \ , \tag{1.3.1a}$$

where

$$\psi(x,y) = \iint A(x,y,p_x,p_y,\tau)\exp\{i\tau\phi(x,y,p_x,p_y)\}dp_x dp_y \ , \tag{1.3.2a}$$

or alternatively,

$$\nabla^2\psi(\overline{r}) + \tau f(x)\psi(\overline{r}) = 0 \ , \tag{1.3.1b}$$

where

$$\psi(\overline{r}) = \iint A(\overline{r},\overline{p},\tau)\exp\{i\tau\phi(\overline{r},\overline{p})\}d\overline{p}, \tag{1.3.2b}$$

with

$$A(x,y,p_x,p_y,\tau) = A(\overline{r},\overline{p},\tau) \sim \sum_k A_k(\overline{r},\overline{p})\tau^{-k} = \sum_k A_k(x,y,p_x,p_y)\tau^{-k} \ ,$$

i.e., $\overline{r} = (x,y)$, $\overline{p} = (p_x,p_y)$.

In Chapter II, phase space analysis is summarized and the Thom potentials are introduced. In Chapter III, $\phi(\overline{r},\overline{p})$ is obtained by an extension of Maslov's technique. This extension allows the determination of $\phi(\overline{r},\overline{p})$ for media characterized by a wide range of depth-dependent profiles. Additionally, this approach enables explicit representation of the emitter geometry and the boundary conditions in $\phi(\overline{r},\overline{p})$ itself. The geometry of the caustic in configuration space, obtained from an analysis of the stationary points of $\phi(\overline{r},\overline{p})$ in phase space, is also determined in Chapter III. In Chapter IV, a transport equation in mixed configuration-momentum space is derived. This transport equation determines the higher order terms in the asymptotic series. To obtain the asymptotic series of the integral in Equation (1.3.2a), alternatively (1.3.2b), requires a coordinate transformation carrying $\phi(\overline{r},\overline{p})$ to an appropriate canonical form. The required canonical form is obtained from a criterion based on Thom's Classification Theorem (Appendix A). In Chapter V, the coordinate transformations are determined explicitly. At regular points in the field, the stationary phase technique determines the asymptotic series of the transformed integrals. At singular stationary points, the classical stationary phase technique must be modified to determine the asymptotic series. In Chapter VI, the asymptotic series at both regular and singular stationary points is determined. Chapter VII concludes this investigation with a summary of the entire algorithm, an explicit example, and suggestions for future research.

CHAPTER II

HEURISTIC SYNOPSIS OF THE THOM POTENTIALS

2.1 Introduction

Most naturally occurring phenomena are nonlinear in character. Often, an exact mathematical analysis of such phenomena is difficult; more often, it is only possible by a linear approximation of the nonlinearity. An example is the plane pendulum, where the nonlinear restoring force at the equilibrium position is approximated as linear to simplify mathematical analysis. It was to avoid such approximations that, in 1881, Poincare [47] introduced the qualitative, or topological, analysis of dynamical systems.

In qualitative dynamics, a dynamical system in configuration space is represented as a field of vectors on phase space. A solution is a smooth curve tangent at each point to the vector based at that point. Qualitative dynamics is concerned with the topological analysis of a family of solution curves throughout the entire phase space, rather than the quantitative analysis of a particular solution curve in configuration space. The family of solution curves in phase space is called the phase portrait.

The topological analysis of a phase portrait leads to a global analysis of the motion. That is, the entire range of motion of the dynamical system can be studied qualitatively based on topological considerations. Regions corresponding to qualitatively different motion, e.g., periodic, aperiodic or asymptotic, are easily determined; and the separatrices, or divides, which demarcate regions of stability

and instability, are readily identified. The principal limitation of this approach is that it does not lead to either a particular explicit solution curve or numerical calculations. Used in conjunction with approximate quantitative methods, such as asymptotic expansions in suitable parameters (also developed by Poincare [48]), it does lead to a numerical analysis of whatever region is selected for study [43].

The phase portrait of a dynamical system is obtained from the configuration space equation of motion. The equation of motion is reduced to an equivalent set of first order differential equations; then an integration determines a solution curve in phase space. The family of these solution curves, each member corresponding to a different initial condition, constitutes the phase portrait. For example, in the second order differential equation,

$$\ddot{x} = F(x,\dot{x})$$

where the dots indicate time derivatives, the substitution of $\dot{x} = y$, $\dot{y} = F(x,y)$ leads to

$$\frac{dy}{dx} = \frac{F(x,y)}{y}$$

An integration determines the solution curve; the family of solution curves is the phase portrait, which is represented pictorially as the graph of $\dot{x}$ vs. x, with direction indicated.

2.2 Structural Stability of Differential Equations

In 1935, Pontryagin and Andronov extended the formulation of Poincare by introducing the concept of structural stability [50]. A dynamical system is said to be structurally stable if, under a small

perturbation, its phase portrait is qualitatively unchanged. For example, if a dynamical system is represented by

$$\begin{aligned} x &= a_1x + b_1y + c_1xy + \ldots \\ y &= a_2x + b_2y + c_2xy + \ldots \end{aligned} , \qquad (2.2.1)$$

the system is structurally stable if the phase portrait is qualitatively unchanged under a small variation of the coefficients.

Under relatively broad assumptions [13], it can be shown that, in a neighborhood of the origin, the analysis of Equations (2.2.1) can be reduced to an analysis of

$$\frac{dy}{dx} = \frac{a_2x + b_2y}{a_1x + b_1y} . \qquad (2.2.2)$$

Heuristically, in an ε-neighborhood of the origin ($|\varepsilon| < 1$), $q^n > q^{n+1}$ ($n \geq 0$), where q is either x or y. Hence, the linear terms are dominant in Equation (2.2.1). The qualitative nature of the phase portrait derives from the eigenvalue structure of

$$S = \begin{bmatrix} a_2 & b_2 \\ a_1 & b_1 \end{bmatrix}$$

That is, the eigenvalues of S may be real, imaginary or complex conjugates; in addition, the real eigenvalues may have the same or different sign, be equal, or one may be zero. The real part of the complex conjugates may be positive or negative. Each eigenvalue structure determines a qualitatively different phase portrait [43]. In this context, strucutral stability refers to how much the coefficients may be perturbed without altering the eigenvalue structure of S.

2.3 Structural Stability of Potential Functions

Structural stability also extends to phenomena which can be characterized by a potential. In this context, structural stability is closely related to the stability of equilibrium points. For a one-dimensional system described by a potential, $V(q)$, the equilibrium point is determined by the condition

$$\frac{dV(q)}{dq} = 0 \ .$$

If $\frac{d^2V}{dq^2} \neq 0$ at the equilibrium point, the equilibrium is *structurally stable*; however, the equilibrium is *dynamically stable* or *unstable* as $\frac{d^2V}{dq^2}$ is *less than* or *greater than zero*. [In this investigation, the historical sign convention for structural stability is followed; hence, the force is defined as $F = V(q)$.] If at the equilibrium position $\frac{d^2V}{dq^2} = 0$, the equilibrium is structurally unstable. That is, an arbitrarily small perturbation of the potential will qualitatively change the phase portrait of the system.

In higher dimensions, the equilibrium point is determined by

$$\nabla V(q) = 0.$$

The equilibrium is stable or unstable depending on the eigenvalue structure of the Hessian matrix, i.e., the eigenvalues of

$$\begin{bmatrix} \dfrac{\partial^2 V}{\partial q_1^{\,2}} & \dfrac{\partial^2 V}{\partial q_1 \partial q_2} & \cdots\cdots & \dfrac{\partial^2 V}{\partial q_1 \partial q_n} \\ \vdots & & & \vdots \\ \dfrac{\partial^2 V}{\partial q_1 \partial q_n} & \cdots\cdots\cdots & & \dfrac{\partial^2 V}{\partial q_n^{\,2}} \end{bmatrix}$$

at the equilibrium point. At dynamically stable equilibrium points, the Hessian determinant is zon-zero and the eigenvalues are all less than zero. If the Hessian determinant is non-zero and the eigenvalues are not all less than zero, the equilibrium is dynamically unstable; but the equilibrium is structurally stable in that an arbitrarily small perturbation will not qualitatively alter the phase portrait of the system. If the Hessian determinant is zero, i.e., the Hessian matrix has at least one vanishing eigenvalue, the equilibrium is structurally unstable. Such a point is called degenerate or singular. If the Hessian matrix is identically zero at the equilibrium position (all eigenvalues zero), the equilibrium is completely degenerate. At structurally stable equilibrium points, it is possible to determine a reversible coordinate transformation carrying $V(\bar{q})$ to a quadratic form [44]. Such a coordinate transformation is called a diffeomorphism. More strictly, a diffeomorphism is a differentiable map with a differentiable inverse. Because the eigenvalue structure is preserved under diffeomorphism [45], the quadratic form is topologically equivalent to $V(\bar{q})$. The condition on the existence of the diffeomorphism is that the Hessian determinant is non-zero at the equilibrium point, i.e., that the Jacobian of $\nabla V(\bar{q})$ is non-zero at the equilibrium point.

2.4 Heuristic Synopsis of the Thom Potentials

2.4.1 Background and introduction. If a function (potential) with a structurally unstable equilibrium point is perturbed, its topology changes. The resulting topology is not always determinable. What Thom did was to determine canonical forms for a class of functions (those capable of parameterization by up to four parameters, e.g., space and time) which could be perturbed into a determinable topology. He sought to study such functions by including them as isolated members of structurally stable families and analyzing their individual behavior through the general behavior of the family. Specifically, Thom's Theorem (Appendix A) classifies functions into one of seven canonical forms based on the degree of degeneracy, i.e., relative structural stability, at the equilibrium point. A complete treatment of the theorem requires rigorous topological analysis, e.g., [18,53]. But classical dynamics does provide a context for a heuristic synopsis, as noted by Haken [29].

2.4.2 One-dimensional potentials (A-series). Consider the potential $V(q,\lambda)$, an analytic function of one dependent variable, q, and an arbitrary physical parameter, λ. For a given λ_o, let $V(q,\lambda_o)$ have an equilibrium point at $q = 0$. Then, the Taylor series of $V(q,\lambda_o)$ about the equilibrium point becomes

$$V(q,\lambda_o) = V(0,\lambda_o) + \frac{1}{2!}\frac{d^2V(0)}{dq^2}q^2 + \frac{1}{3!}\frac{d^3V(0)}{dq^3}q^3 + \ldots + \frac{1}{n!}\frac{d^nV(0)}{dq^n}q^n + \ldots$$

The equilibrium is dynamically stable or unstable as the sign of $\frac{d^2V(0)}{dq^2}$ is negative or positive. If, for $\lambda = \lambda_o$, $\frac{d^2V(0)}{dq^2} = 0$, the equilibrium

point is not structurally stable, i.e., some arbitrarily small perturbation in λ could lead to either a stable or unstable equilibrium.

Let the parameter λ be such that at $\lambda = \lambda_o$

$$\frac{dV(0)}{dq} = \frac{d^2V(0)}{dq^2} = 0$$

and

$$\frac{d^3V(0)}{dq^3} \neq 0 \ .$$

Then, in the ε-neighborhood of the origin, the coordinate transformation

$$\xi = q\left[\frac{1}{3!}\frac{d^3V(0)}{dq^3} + \frac{1}{4!}\frac{d^4V(0)}{dq^4}q + \ldots + \frac{1}{n!}\frac{d^nV(0)}{dq^n}q^{n-3} + \ldots\right]^{1/3}$$

transforms $V(q,\lambda_o)$ precisely to

$$V(\xi,\lambda_o) = V(0,\lambda_o) + \xi^3 \ , \qquad (2.4.2.1)$$

with the coefficient of ξ^3 normalized to unity. (Heuristically, in an ε-neighborhood of the origin, $q^n >> q^{n+1}$; hence, the potential can be closely approximated by its lowest order term.) If $V(0,\lambda_o) = 0$, $V(\xi,\lambda_o) = \xi^3$; this is the simplest single-variable degeneracy.

If λ is varied slightly to $\tilde{\lambda}$, the potential becomes

$$V(\xi,\tilde{\lambda}) = V(0,\tilde{\lambda}) + \alpha\xi + \beta\xi^2 + \xi^3 + \delta\xi^4 + \ \ldots \ , \qquad (2.4.2.2)$$

where the coefficient of ξ^3 is normalized to unity. For sufficiently small λ-variations in an ε-neighborhood of the origin, $V(\xi,\tilde{\lambda})$ may be closely approximated by

$$V(\xi,\tilde{\lambda}) = V(0,\tilde{\lambda}) + \alpha\xi + \beta\xi^2 + \gamma\xi^3 \ , \tag{2.4.2.3}$$

which is qualitatively different from Equation (2.4.2.1) because it is nondegenerate, i.e., structurally stable, at its equilibrium point. Equation (2.4.2.3) may be simplified by translating the origin to $-\beta/3$ and shifting the zero-point of the potential to $2\beta^2/27 - \alpha\beta/3 + V(0,\tilde{\lambda})$. The resulting form is

$$V(\xi,\tilde{\lambda}) = \xi^3 + \kappa\xi \ , \tag{2.4.2.4}$$

where $\kappa = \alpha - \beta^2/9$. Equation (2.4.2.4) is the form of the lowest order Thom potential. It represents a structurally stable family of potentials parameterized by κ. The structurally unstable potential in Equation (2.4.2.1) appears as an isolated member when $\kappa = 0$ and the zero-point is appropriately shifted. [Thom sought to study the behavior of Equation (2.4.2.1) through an analysis of the behavior of Equation (2.4.2.4) as $\kappa \to 0$.] Those perturbations which qualitatively change the original degeneracy are called unfoldings. Because all possible unfoldings of ξ^3 near the origin can be transformed into $\kappa\xi$, it is called the universal unfolding. [While this treatment has been heuristic, Thom's Theorem states that Equation (2.4.2.2) may be transformed into Equation (2.4.2.4) exactly.]

In one dimension, there exist Thom potentials corresponding to q^4, q^5 and q^6 degeneracies in Equation (2.4.2.1). An analysis similar to the above determines the universal unfoldings. These one-dimensional potentials are called the A-series and are listed in Appendix A.

2.4.3 Two-dimensional potentials. Consider the potential $V(q_1,q_2,\lambda)$, an analytic function of two independent variables, q_1 and q_2, and an arbitrary physical parameter, λ. For a given $\lambda = \lambda_o$, the equilibrium point, $\bar{q}_o = (q_{1o}, q_{2o})$, is determined from

$$\nabla V(q_1,q_2,\lambda_o) = 0 \ .$$

When the Hessian determinant is non-zero at $\bar{q}_o$, the equilibrium is structurally stable. If the Hessian determinant is zero at $\bar{q}_o$ (i.e., the Hessian has at least one vanishing eigenvalue), the equilibrium is not structurally stable. When the Hessian has one vanishing eigenvalue, the system may be considered to lose structural stability in one direction for, under a principal axis transformation, one homogeneous quadratic and the cross-term, i.e, q_1q_2, in the Taylor expansion of $V(q_1,q_2,\lambda_o)$ at $\bar{q}_o$ will vanish. The degree of degeneracy is determined by the degree of the first non-vanishing Taylor series term in the direction of structural instability.

For example, let $V(q_1,q_2,\lambda_o)$ have a degeneracy at the origin corresponding to q^3 above. Let $\alpha(\alpha > 0$, for clarity) be the non-zero eigenvalue of the Hessian and let the eigenvectors be

$$\bar{e}_\alpha = \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} \qquad \bar{e}_o = \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix}$$

Then, the principal axis transformation

$$q_1 = \xi_1 e_{11} + \xi_2 e_{12}$$

$$q_2 = \xi_1 e_{21} + \xi_2 e_{22}$$

carries $V(q_1,q_2,\lambda_o)$ at $q=(\overline{0})$ to

$$\overline{V}(\xi_1,\xi_2,\lambda_o)=V(\overline{0},\lambda_o)+g(\xi_1,\xi_2)\xi_1^{\,2}+2\xi_1\xi_2^{\,2}h(\xi_2)+\ell(\xi_2)\xi_2^{\,3} ,$$

with $g(\overline{0})=\alpha$, $\ell(0)\neq 0$. Then, the coordinate transformation

$$\beta_1=g(\xi_1,\xi_2)^{1/2}\xi_1+g(\xi_1,\xi_2)^{-1/2}h(\xi_2)\xi_2^{\,2}$$

$$\beta_2=[\ell(\xi_2)-g(\xi_1,\xi_2)^{-1}h(\xi_2)^2\xi_2]^{1/3}\xi_2 ,$$

where the negative exponent refers to powers rather than inverses, determines precisely

$$\tilde{V}(\beta_1,\beta_2,\lambda_o)=V(\overline{0},\lambda_o)+\beta_1^{\,2}+\beta_2^{\,3} .$$

$\tilde{V}(\beta_1,\beta_2,\lambda_o)$ may be considered the canonical form for the degeneracy. The universal unfolding in the β_2 direction proceeds as in the one-dimensional case. The analogous canonical form is, therefore

$$\tilde{\tilde{V}}(\beta_1,\beta_2,\lambda_o) = \beta_1^{\,2}+\beta_2^{\,3}+\kappa\beta_2 .$$

When the Hessian matrix of $V(q_1,q_2,\lambda_o)$ at $\overline{q}=(\overline{0})$ is itself zero, the topology of the phase portrait is determined by the cubic terms in the Taylor expansion of $V(q_1,q_2,\lambda_o)$ about $\overline{q}=(\overline{0})$

$$V^3(\overline{0},\lambda_o)=c_1q_1^{\,3}+c_2q_1^{\,2}q_2+c_3q_1q_2^{\,2}+c_4q_2^{\,3} , \qquad (2.4.3.1)$$

where $V^3(\overline{0},\lambda_o)$ represents the third derivative term with the c_i the appropriate Taylor coefficients. If $V^3(\overline{0},\lambda_o)\neq 0$, and at least one of c_1 or c_4 is non-zero, the cubic is equivalent to one of four canonical forms. The equivalence is determined by an analysis of the root

structure of the cubic, i.e., the number of real roots of $V^3(\bar{0},\lambda_o) = 0$ and whether or not any of the roots are equal. Each canonical form is equivalent to a Thom potential.

To determine the root structure of Equation (2.4.3.1), $V^3(\bar{0},\lambda_o)$ is equated to zero, then reduced to a one-dimensional form by dividing through by either q_1^3 or q_2^3 to yield

$$c_1t^3 + c_2t^2 + c_3t + c_4 = 0 \ , \qquad (2.4.3.2)$$

where $t = q_1/q_2$, with $c_1, c_4 \neq 0$ for generality. Equation (2.4.3.2) can have four root structures: three real equal roots, three real roots with two equal, three unequal roots, or one real root and one complex - conjugate pair. Linear transformations have been determined which carry each root structure to the canonical forms [49]:

(i)	three real equal roots	$:x^3$
(ii)	three real unequal roots	$:x^3 - xy^2$
(iii)	one real root and one complex conjugate pair	$:x^3 + y^3$
(iv)	three real roots, with two equal	$:x^2y$.

x^3 is the lowest order Thom potential. When associated with a one-dimensional dynamical system, x^3 may be perturbed into the unfolding $\xi^3 + \kappa\xi$ [Equation (2.4.2.2)]. When associated with a two-dimensional dynamical system, a perturbation of x^3 may lead to any of an infinite number of unfoldings, i.e., an indeterminate topology. Such potentials are qualitatively different from those considered by Thom and are not included in his Classification Theorem. $x^3 - xy^2$ and $x^3 + y^3$

lead to the unfoldings which are called the elliptic and hyperbolic unfoldings, respectively. x^2y leads to the form x^2y+y^4, which, in turn, leads to the unfolding called the parabolic umbilic.

If, in Equation (2.4.3.1), $c_1,c_4=0$ and $c_2,c_3\neq 0$, the cubic corresponds to the form x^2y. If c_1,c_4 and one of c_2 or c_3 equal zero, there is no equivalent Thom potential.

The equilibrium point of the elliptic umbilic, $\phi_E(x,y,\lambda_o)=x^3-xy^2$, is determined from

$$\nabla\phi_E(x,y,\lambda_o)=0 ,$$

specifically

$$(3x^2-y^2)\hat{i}-2xy\hat{j}=0 ,$$

to be $(x,y)=(\overline{0})$. At $(x,y)=(\overline{0})$, the Hessian matrix of $\phi_E(x,y,\lambda_o)$ is identically zero; hence, it is completely degenerate.

If λ is varied slightly, e.g., $\lambda_o\to\tilde{\lambda}$, in general, $\phi_E(x,y,\lambda_o)$ becomes in an ε-neighborhood of the origin

$$\phi_E(x,y,\tilde{\lambda})=k_1+k_2x+k_3y+k_4x^2+k_5y^2+k_6xy+x^3-xy^2 , \qquad (2.4.3.3)$$

where the coefficient of x^3-xy^2 has been normalized to unity. Then, by translating the origin to $x=x'-k_4/3$, $y=y'-k_5/3$ and shifting the zero-point of the potential to

$$k_1+\frac{2}{27}k_4{}^3+\frac{k_5{}^2}{9}+\frac{k_4k_5{}^2}{27}+\frac{k_4k_5}{9}+k_3k_5,$$

Equation (2.4.3.3) is transformed to

$$\tilde{\phi}_E(x,y,\tilde{\lambda})=x^3-xy^2+\kappa_1y^2+\kappa_2x+\kappa_3y . \qquad (2.4.3.4)$$

Equation (2.4.3.4) is the canonical form of the elliptic umbilic.

An analysis of the hyperbolic umbilic, $\phi_E(x,y,\lambda_o) = x^3 + y^3$, proceeds similarly. The equilibrium point is again $(x,y) = (\overline{0})$; at $(x,y) = (\overline{0})$, the Hessian matrix of $\phi_H(x,y,\lambda_o)$ is identically zero. If λ is varied slightly to $\tilde{\lambda}$, in an ε-neighborhood of the origin, $\phi_H(x,y,\lambda_o)$ becomes

$$\phi_H(x,y,\tilde{\lambda}) = k_1 + k_2x + k_3y + k_4x^2 + k_5y^2 + k_6xy + x^3 + y^3 \ , \qquad (2.4.3.5)$$

where the coefficient of $x^3 + y^3$ has been normalized to unity. Then, by an axis translation and a shift of the zero point of the potential, Equation (2.4.3.5) becomes

$$\tilde{\phi}_H(x,y,\tilde{\lambda}) = x^3 + y^3 + \kappa_1 xy + \kappa_2 x + \kappa_3 y \ , \qquad (2.4.3.6)$$

which is the canonical form of the hyperbolic umbilic.

The form $\phi_p(x,y,\lambda_o) = x^2y$ cannot be treated similarly, for applying the equilibrium condition does not determine a unique equilibrium point. That is, setting

$$\nabla\phi_p(x,y,\lambda_o) = 0$$

yields

$$2xy\hat{i} + x^2\hat{j} = 0 .$$

At $x = 0$, the condition is satisfied, but the y-coordinate is not determined uniquely. Hence, $\phi_p(x,y,\lambda_o)$ must be perturbed into an equivalent form which determines a unique equilibrium point. Because the indeterminacy is in the y-coordinate, a perturbation along the

y-axis seems heuristically appropriate. Adding $\pm y$ to $\pm y^2$ to $\phi_p(x,y,\lambda_o)$ uniquely determines the equilibrium point but also obtains Hessian matrices which are not completely degenerate; adding $\pm y$ or $\pm y^2$ renders the potential structurally stable. Adding $+y^3$ leads to x^2y+y^3, an alternate form of the hyperbolic umbilic [18]. Adding $-y^3$ leads to x^2y-y^3, the form of the elliptic umbilic. But adding y^4 to $\phi_p(x,y,\lambda_o)$ determines

$$\tilde{\phi}(x,y,\lambda_o) = x^2y+y^4 \ .$$

At $(x,y) = (\bar{0})$, the Hessian matrices of both $\phi_p(x,y,\lambda_o)$ and $\tilde{\phi}_p(x,y,\lambda_o)$ are both completely degenerate; however, at $(x,y) = (\bar{0})$, $\tilde{\phi}_p(x,y,\lambda_o)$ has a unique equilibrium. In an ε-neighborhood of the origin, $|x^2y| > |y^4|$; thus, $\tilde{\phi}_p(x,y,\lambda_o)$ may be regarded as a tractable perturbation of $\phi_p(x,y,\lambda_o)$. $\tilde{\phi}_p(x,y,\lambda_o) = x^2y+y^4$ leads to the unfolding called the parabolic umbilic. (A completely rigorous treatment of why x^2y leads to the form x^2y+y^4 requires an analysis based on topological considerations, e.g. [18,35].) Thus, proceeding as with the elliptic and hyperbolic umbilics, we determine the form

$$\bar{\phi}_p(x,y,\tilde{\lambda}) = x^2y+y^4+\kappa_1x^2+\kappa_2y^2+\kappa_3x+\kappa_4y \ , \qquad (2.4.3.7)$$

which is the canonical form of the parabolic umbilic.

If, in Equation (2.4.3.1), $V^3(\bar{q},\bar{\lambda}_o) \equiv 0$, the degeneracy must be studied from quartic, or higher, terms in the Taylor expansion. Such degeneracies are qualitatively different from those considered above and are not included in the Thom classification.

This, admittedly, heuristic synopsis represents an amplification of an approach outlined by Haken [29]. The results of this chapter have been rigorously proven in a series of papers by Mather [36-41].

CHAPTER III

DETERMINATION OF THE PHASE

3.1 Introduction

The determination of the phase, $\phi(\bar{r},\bar{p})$, and its relation to the medium profile, is crucial to the investigation of the field represented by the integral

$$\psi(\bar{r}) = \int A(\bar{r},\bar{p},\tau)\exp\{i\tau\phi(\bar{r},\bar{p})\}d\bar{p} , \tag{1.3.2b}$$

where

$$A(\bar{r},\bar{p},\tau) \sim \sum_k A_k(\bar{r},\bar{p})\tau^{-k} .$$

In the mixed configuration-momentum space approach, the phase is especially important. Not only are the stationary points of the phase obtained from $\phi(\bar{r},\bar{p})$, but the geometry of the caustic as well. The standard approach is to determine $\phi(\bar{r},\bar{p})$ from the eikonal Equation (1.2.5) and the $A_k(\bar{r},\bar{p})$ from the transport Equation (1.2.6) in momentum space [34]. As in configuration space, in momentum space, these are first order nonlinear partial differential equations.

In this chapter, the phase is determined for a medium described by a single variable profile through an extension of Maslov's approach. This procedure will allow an explicit characterization of the emitter geometry and the boundary conditions in the phase itself. The geometry of the caustic in configuration space is then determined from an analysis of the stationary points of $\phi(\bar{r},\bar{p})$ in phase space.

3.2 The Technique of Maslov

Maslov considered the integral representation for the field, Equation (1.3.2b). But rather than determine the phase from the

eikonal, he showed $\phi(\bar{r},\bar{p})$ had the form

$$\phi(\bar{r},\bar{p}) = \bar{r}\cdot\bar{p} - S(\bar{p}) \ , \tag{3.2.1}$$

where $S(\bar{p})$ is the generating function of a canonical transformation [3]. From Equation (3.2.1), the stationary phase condition, $\nabla_p\phi(\bar{r},\bar{p}) = 0$, becomes

$$\bar{r} - \nabla_p S(\bar{p}) = 0, \tag{3.2.2}$$

i.e., a Lagrange manifold. A Lagrange manifold may be defined as a surface determined by the gradient of a generating function. By rewriting Equation (3.2.2) in component form, i.e.,

$$\begin{aligned} x - \frac{\partial S(\bar{p})}{\partial p_x} &= 0 \\ y - \frac{\partial S(\bar{p})}{\partial p_y} &= 0 \ , \end{aligned} \tag{3.2.3}$$

the Lagrange manifold may be seen as a transformation from momentum space to configuration space. On the Lagrange manifold, i.e., for $\bar{r} = \nabla_p S(\bar{p})$, the eikonal condition in Equation (1.2.5) becomes

$$\bar{p}\cdot\bar{p} - f(\bar{r}) = 0 \ , \tag{3.2.4a}$$

or more explicitly

$$\bar{p}\cdot\bar{p} - f[\nabla_p S(\bar{p})] = 0 \ . \tag{3.2.4b}$$

Maslov proceeded by defining his Hamiltonian as

$$H = \bar{p}\cdot\bar{p} - f(\bar{r}) \ , \tag{3.2.5}$$

for arbitrary $(\bar{r},\bar{p})$. Because the Lagrange manifold is invariant

under the Hamiltonian flow determined from Equation (3.2.5), i.e., a Hamiltonian trajectory satisfying Equation (3.2.2) for $t = 0$ satisfies Equation (3.2.2) for all t [10], $S(\overline{p})$ may always be determined, e.g., using Hamilton-Jacobi theory [31].

For example, in two dimensions $S(\overline{p})$ may be determined by applying Hamilton's equations to Equation (3.2.5), leading to

$$\dot{x} = 2p_x \qquad \dot{p}_x = \frac{\partial f(x,y)}{\partial x}$$

$$\dot{y} = 2p_y \qquad \dot{p}_y = \frac{\partial f(x,y)}{\partial y}$$

which obtains the map

$$\begin{aligned} x &= x(t,\Theta) \qquad p_x = p_x(t,\Theta) \\ y &= y(t,\Theta) \qquad p_y = p_y(t,\Theta) \ , \end{aligned} \tag{3.2.6}$$

where t is the time and Θ is a parametrized initial condition. Then inverting the momentum equations in Equation (3.2.6) yields

$$\begin{aligned} t &= t(p_x,p_y) \\ \Theta &= \Theta(p_x,p_y). \end{aligned} \tag{3.2.7}$$

Substituting Equation (3.2.7) into the configuration space equations in Equation (3.2.6) determines

$$\begin{aligned} x &= x[t(p_x,p_y),\ \Theta(p_x,p_y)] = X(p_x,p_y) \\ y &= y[t(p_x,p_y),\ \Theta(p_x,p_y)] = Y(p_x,p_y) \ , \end{aligned} \tag{3.2.8}$$

where $X(p_x,p_y)$ and $Y(p_x,p_y)$ are explicit functions of p_x and p_y. Since the Hamiltonian flow preserves the Lagrange manifold, from Equations (3.2.3) and (3.2.8),

$$\frac{\partial S(\bar{p})}{\partial p_x} = X(p_x,p_y)$$
$$\frac{\partial S(\bar{p})}{\partial p_y} = Y(p_x,p_y) \; . \tag{3.2.9}$$

Integrating Equations (3.2.9) determines $S(\bar{p})$ which, in turn, determines $\phi(\bar{r},\bar{p})$, [28].

3.3 An Extension of Maslov's Technique

For those media characterized by a single variable profile, $f(x)$, an extension of this approach leads to a phase which includes an explicit representation of the emitter geometry and the boundary conditions on the wave vectors.

Consider the Hamiltonian (on the Lagrange manifold)

$$H = p_x^2 + p_y^2 - f(x) = 0 \; . \tag{3.3.1}$$

Then, at any field point where f^{-1} is defined,

$$x = f^{-1}(p_x^2 + p_y^2) \tag{3.3.2}$$

determines one coordinate of the Lagrange manifold

$$x = \frac{\partial S(\bar{p})}{\partial p_x} \; . \tag{3.3.3}$$

The other spatial coordinate may be determined by noting that, for gradient (C^2) functions,

$$\frac{\partial^2 S(\bar{p})}{\partial p_x \partial p_y} = \frac{\partial^2 S(\bar{p})}{\partial p_y \partial p_x} \; ;$$

hence,

$$y = \int \frac{\partial^2 S(\bar{p})}{\partial p_x \partial p_y} \, dp_x = \frac{\partial \tilde{S}(\bar{p})}{\partial p_y} + \Theta(p_y) = \frac{\partial S(\bar{p})}{\partial p_y} \; , \tag{3.3.4}$$

where $\Theta(p_y)$ is an arbitrary function of p_y. Thus,

$$x = \frac{\partial S(\bar{p})}{\partial p_x} = f^{-1}(p_x{}^2 + p_y{}^2)$$

$$y = \frac{\partial S(\bar{p})}{\partial p_y} = \frac{\partial \tilde{S}(\bar{p})}{\partial p_y} + \Theta(p_y)$$

define the Lagrange manifold, where $\Theta(p_y)$ derives from the boundary conditions at the emitter.

Let $\Theta(p_y)$ be represented by a polynomial in p_y of arbitrary power with constant coefficients, and let the geometry of the emitter be represented by

$$y = g(x) \ . \qquad (3.3.5)$$

The spatial coordinates of the emitter taken together with the initial boundary conditions on the wave vectors may be regarded as a lifting from configuration space to the mixed configuration-momentum space, i.e., from 2-space to 4-space. Then, by choosing the intital conditions on the emitter so that, at $t = 0$, $[\bar{r}(t), \bar{p}(t)]$ satisfy Equation (3.3.2), Equations (3.3.3), (3.3.4) and (3.3.5) determine

$$g(x) = \frac{\partial \tilde{S}(\bar{p})}{\partial p_y} + a_o + a_1 p_y + \ldots = \frac{\partial S(\bar{p})}{\partial p_y} \ , \qquad (3.3.6)$$

on the emitter. Solving Equation (3.3.3) for p_x and substituting into Equation (3.3.6) determines an equation in x and p_y. Successive differentiations with respect to x lead to a system of linear algebraic equations for the a_i in terms of the initial wave vectors and the spatial variations of the wave vectors at a point on the emitter. Solving for the a_i determines Equation (3.3.4) explicitly. Equations (3.3.3) and (3.3.4) determine the Lagrange manifold, and hence,

$$\phi(\bar{r},\bar{p}) = \phi(x,y,p_x,p_y) = xp_x + yp_y - S(p_x,p_y) \ . \tag{3.3.7}$$

3.4 Determination of the Caustic Geometry

The analysis of the field at any point $\bar{r}_o = (x_o,y_o)$ also proceeds from the phase. The Lagrange manifold conditions at $\bar{r}_o$ are solved for the associated momentum, $\bar{p}_o = (p_{xo},p_{yo})$. If the Hessian determinant of $\phi(\bar{r},\bar{p})$ at $(\bar{r}_o,\bar{p}_o)$, i.e.,

$$\det \begin{bmatrix} \dfrac{\partial^2\phi}{\partial p_x^{\,2}} & \dfrac{\partial^2\phi}{\partial p_x \partial p_y} \\ \dfrac{\partial^2\phi}{\partial p_y \partial p_x} & \dfrac{\partial^2\phi}{\partial p_y^{\,2}} \end{bmatrix}_{(\bar{r}_o,\bar{p}_o)} ,$$

is non-zero, $(\bar{r}_o,\bar{p}_o)$ is a regular point. The non-vanishing of the Hessian determinant at a stationary point is analogous to the non-vanishing of the second derivative of the phase in one dimension (Chapter I). Hence, it is the condition for the validity of the classical stationary phase technique in higher dimensions [15]. If the Hessian determinant of $\phi(\bar{r},\bar{p})$ at $(\bar{r}_o,\bar{p}_o)$ is zero, i.e., $\phi(\bar{r},\bar{p})$ is structurally unstable at $(\bar{r}_o,\bar{p}_o)$, $(\bar{r}_o,\bar{p}_o)$ is a singular point. Consequently, the classical stationary phase technique would not be valid at $(\bar{r}_o,\bar{p}_o)$.

The equation of the caustic may then be determined from the Hessian. Equating the Hessian determinant to zero defines a curve of singular points in momentum space, i.e., the caustic in momentum space. Associated with every point $\bar{p}_o$ on the caustic is a point $\bar{r}_o = (x_o,y_o)$ in configuration space determined by the Lagrange manifold condition

$$x_o = \frac{\partial S(\bar{p}_o)}{\partial p_x} \qquad y_o = \frac{\partial S(\bar{p}_o)}{\partial p_y} \quad .$$

The locus of these points determines the equation of the caustic in configuration space.

CHAPTER IV

THE TRANSPORT EQUATION

4.1 Introduction

In this investigation, the reduced Helmholtz wave equation

$$\nabla^2\psi(\overline{r}) + \tau f(x)\psi(\overline{r}) = 0 \ , \tag{1.3.1b}$$

where τ is a large parameter, has an asymptotic solution of the form

$$\psi(\overline{r}) - \int A(\overline{r},\overline{p},\tau)\exp\{i\tau\phi(\overline{r},\overline{p})\}d\overline{p} = O(\tau^{-\infty}) \ , \tag{4.1.1}$$

where

$$A(\overline{r},\overline{p},\tau) \sim \sum_k A_k(\overline{r},\overline{p})\tau^{-k} \ ,$$

which is an alternate form of Equation (1.3.2b), [22,23]. Ordinarily, the phase is determined from the eikonal Equation (1.2.5) and the series coefficients are determined from the transport Equation (1.2.6). Because the eikonal and transport equations have representations in both configuration space and momentum space, it is possible to determine the phase and the series coefficients exclusively in either space [24]. Here, following Maslov and Duistermaat [22,23], we consider a mixed configuration-momentum space representation for the phase, $\phi(\overline{r},\overline{p})$, and the amplitude, $A(\overline{r},\overline{p},\tau)$. The phase has been determined without recourse to the eikonal equation (Chapter III). To determine the series coefficients, $A_k(\overline{r},\overline{p})$, it is necessary to derive an equation in the mixed space representation that is analogous to the transport equation.

4.2 Derivation of the Modified Transport Equation

Let the asymptotic solution of Equation (4.1.1) be represented as

$$\psi(\bar{r}) - \int A(\bar{r},\bar{p},\tau)\exp\{i\tau[\bar{r}\cdot\bar{p} - S(\bar{p})]\}d\bar{p} = O(\tau^{-\infty}) \;, \tag{4.2.1}$$

i.e., the form of the phase (from Chapter III) is explicitly noted. The derivation of the transport equation proceeds by carrying the differentation in Equation (1.3.1b) across the integral in Equation (4.2.1), obtaining

$$\begin{aligned}&\nabla^2\psi(\bar{r}) + \tau f(x)\psi(\bar{r})\\ &\quad - \int \exp\{i\tau[\bar{r}\cdot\bar{p} - S(\bar{p})]\}[(i\tau)^2(\bar{p}\cdot\bar{p} - f(x))A + 2i\tau(\bar{p}\cdot\nabla_r A)\\ &\quad + \nabla_r{}^2 A]d\bar{p} = O(\tau^{-\infty}) \;,\end{aligned} \tag{4.2.2}$$

where, for clarity, the argument of $A(\bar{r},\bar{p},\tau)$ is included implicitly. Expanding $\bar{p}\cdot\bar{p} - f(x)$ in a Taylor series about the turning point determines

$$\bar{p}\cdot\bar{p} - f(x) = \bar{p}\cdot\bar{p} - f(\partial p_x S) + [\bar{r} - \nabla_p S(\bar{p})]\cdot\bar{D}(\bar{r},\bar{p}) \;, \tag{4.2.3}$$

where $\bar{D}(\bar{r},\bar{p})$ is the remainder term, which may be represented as

$$\bar{D}(\bar{r},\bar{p}) = \bar{D} = -\int_0^1 \nabla_r f[t(\bar{r}-\nabla_p S) + \nabla_p S]dt \;.$$

The first term is precisely Maslov's Hamiltonian [and thus is zero by Equation (3.2.6)] on the Lagrange manifold; hence, Equation (4.2.3) becomes

$$\bar{p}\cdot\bar{p}-f(x)=[\bar{r}-\nabla_p S(\bar{p})]\cdot\bar{D}(\bar{r},\bar{p}) \quad . \tag{4.2.4}$$

Substituting Equation (4.2.4) into Equation (4.2.2) determines

$$\begin{aligned}&\nabla^2\psi(\bar{r})+\tau f(x)\psi(\bar{r})\\ &\quad-\int \exp\{i\tau[\bar{r}\cdot\bar{p}-S(\bar{p})]\}[(i\tau)^2[\bar{r}-\nabla_p S]\cdot\bar{D}A+2i\tau(\bar{p}\cdot\nabla_r A)\\ &\quad+\nabla_r{}^2 A]d\bar{p}=0(\tau^{-\infty}) \quad .\end{aligned} \tag{4.2.5}$$

But

$$\begin{aligned}&\int \exp\{i\tau[\bar{r}\cdot\bar{p}-S(p)]\}(i\tau)^2[\bar{r}-\nabla_p S(\bar{p})]\cdot\bar{D}Ad\bar{p}\\ &\quad= i\tau\int \nabla_p\cdot(\exp\{i\tau[\bar{r}\cdot\bar{p}-S(\bar{p})]\}A\bar{D})d\bar{p}\\ &\quad- i\tau\int \exp\{i\tau[\bar{r}\cdot\bar{p}-S(\bar{p})]\}[\bar{D}\cdot\nabla_p A+A\nabla_p\cdot\bar{D}]d\bar{p} \quad .\end{aligned} \tag{4.2.6}$$

By substituting Equation (4.2.6) into Equation (4.2.5) and taking the surface integral over a sufficiently large radius that it vanishes, Equation (4.2.5) becomes

$$\begin{aligned}&\nabla^2\psi(\bar{r})+\tau f(x)\psi(\bar{r})\\ &\quad-\int \exp\{i\tau[\bar{r}\cdot\bar{p}-S(\bar{p})]\}i\tau[-\bar{D}\cdot\nabla_p A-A\nabla_p\cdot\bar{D}+2\bar{p}\cdot\nabla_r A\\ &\quad+\frac{1}{i\tau}\nabla_r{}^2 A]d\bar{p}=0(\tau^{-\infty}) \quad .\end{aligned} \tag{4.2.7}$$

For Equation (4.2.1) to have an asymptotic solution, it is necessary that Equation (4.2.7) be satisfied in a neighborhood of the Lagrange manifold. Noting Equation (1.3.1b), a sufficient condition for Equation (4.2.7) is that

$$-\overline{D} \cdot \nabla_p A - A\nabla_p \cdot \overline{D} + 2\overline{p} \cdot \nabla_r A + \frac{1}{i\tau} \nabla_r^{\,2} A = 0. \tag{4.2.8}$$

Then by introducing the flow

$$\begin{aligned} \dot{\overline{r}} &= 2\overline{p} \\ \dot{\overline{p}} &= -\overline{D}(\overline{r},\overline{p}), \end{aligned} \tag{4.2.9}$$

Equation (4.2.8) becomes

$$\dot{A}_k - A_k \nabla_p \cdot \overline{D} + \nabla_r^{\,2} A_{k-1} = 0 \ , \tag{4.2.10}$$

where the dots indicate the time derivatives determined by the flow in Equation (4.2.9). Equation (4.2.10) determines the evolution of the A_k's as does the transport equation (in either configuration or momentum space [24]). Hence, we regard Equation (4.2.10) as the transport equation in the mixed configuration-momentum space representation. [It should be noted that if the flow is defined by the Hamiltonian, i.e.,

$$\dot{\overline{r}} = 2\overline{p}$$

$$\dot{\overline{p}} = \nabla_x f(x) \ ,$$

it is only possible to determine the asymptotic limit, i.e., $A_o(\overline{r},\overline{p})$.]

4.3 Outline of the Algorithm

The basic algorithm can now be outlined. At any point, $r_o = (x_o, y_o)$, in the field the phase is determined by an extension of Maslov's technique (Chapter III). Next, $A_o(\overline{r},\overline{p})$ is obtained from the transport equation. Then, the asymptotic series of the integral

$$\int A_o(\overline{r}_o,\overline{p}) \exp\{i\tau\phi(\overline{r}_o,\overline{p})\} d\overline{p}$$

is determined. Additional $A_k(\overline{r}_o,\overline{p})$ proceed from Equation (4.2.10), yielding the integrals

$$\int A_k(\overline{r}_o,\overline{p})\exp\{i\tau\phi(\overline{r}_o,\overline{p})\}d\overline{p} \ .$$

Each integral determines an asymptotic series. The entire asymptotic series at $\overline{r}_o$ is the sum of the series of the integrals.

CHAPTER V

THE COORDINATE TRANSFORMATIONS

5.1 Introduction

We consider an integral of the form

$$I = \iint A(x,y,p_x,p_y,\tau)\exp\{i\tau\phi(x,y,p_x,p_y)\}dp_x dp_y \ , \qquad (1.3.2a)$$

where $A(x,y,p_x,p_y,\tau)$ and all its derivatives are bounded throughout the region of integration, $A(x,y,p_x,p_y,\tau)$ is analytic near the stationary points of $\phi(x,y,p_x,p_y)$ and $\phi(x,y,p_x,p_y)$ has no poles in the region of integration. Before the asymptotic series of Equation (1.3.2a) can be determined, the integral must be transformed to a form appropriate for asymptotic analysis, i.e., the phase must be transformed to a canonical form. The transformation proceeds on the nature of the Hessian determinant of the phase, $\phi(x,y,p_x,p_y)$, at the stationary point, $(\bar{r}_o,\bar{p}_o) = (x_o,y_o,p_{xo},p_{yo})$. If the Hessian determinant is non-zero at the stationary point, $\phi(x,y,p_x,p_y)$ at $(\bar{r}_o,\bar{p}_o)$ is transformed to

$$\tilde{\phi}(\bar{r}_o,\beta_1,\beta_2) = \phi(\bar{r}_o,\bar{p}_o) \pm \beta_1^{\,2} \pm \beta_2^{\,2} \ , \qquad (5.1.1)$$

which follows from the Morse Lemma [44]. If the Hessian determinant is zero at the stationary point with one vanishing eigenvalue, then the phase is transformed to

$$\tilde{\phi}(\bar{r}_o,\beta_1,\beta_2) = \phi(\bar{r}_o,\bar{p}_o) \pm \beta_1^{\,2} \pm \beta_2^{\,n} \ , \qquad (5.1.2)$$

which follows from the Gromoll-Meyer Lemma [27]. The exponent of β_2 is determined from the Thom potential to which $\phi(\bar{r},\bar{p})$ at $(\bar{r}_o,\bar{p}_o)$

is equivalent (Appendix A). When the Hessian determinant at the stationary point has two vanishing eigenvalues, the transformed potential assumes a qualitatively different form than in Equations (5.1.1) and (5.1.2). This case is not considered here. The transformation of the phase to a canonical form allows the integral in Equation (1.3.2a) to be expressed in a form suitable for asymptotic analysis. In this chapter, we determine the required coordinate transformations.

5.2 Regular Stationary Points

At $\overline{r}_o = (x_o, y_o)$, let $\phi(x,y,p_x,p_y) = \phi(\overline{r},\overline{p})$ have a stationary point $[\overline{p}_o = (p_{xo}, p_{yo})$ such that $\nabla_p \phi(\overline{r}_o, \overline{p}_o) = 0$, taken here at $\overline{p}_o = (\overline{0})$ for clarity]. Then, the Taylor series of $\phi(\overline{r},\overline{p})$ at $(\overline{r}_o,\overline{p}_o)$ is

$$\phi(\overline{r},\overline{p}) = \phi(\overline{r}_o,\overline{0}) + C_1(p_x)p_x^2 + 2p_x C_2(p_x,p_y) + C_3(p_y)p_y^2 , \qquad (5.2.1)$$

where $C_1(0)$ and $C_3(0) \neq 0$. Let

$$\overline{e}_{\lambda 1} = \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix}, \qquad \overline{e}_{\lambda 2} = \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix}$$

be the eigenvectors corresponding to the eigenvalues at λ_1 and λ_2, respectively. Then, the principal axis transformation

$$\begin{aligned} p_x &= \xi_1 e_{11} + \xi_2 e_{12} \\ p_y &= \xi_1 e_{21} + \xi_2 e_{22} \end{aligned} \qquad (5.2.2)$$

and Taylor's Theorem carry $\phi(\overline{r},\overline{p})$ at $(\overline{r}_o,\overline{p}_o)$ to

$$\overline{\phi}(\overline{r}_o,\xi_1,\xi_2) = \phi(\overline{r}_o,\overline{0}) + g_1(\xi_1,\xi_2)\xi_1^2 + 2\xi_1\xi_2^2 h(\xi_2) + g_2(\xi_2)\xi_2^2 ,$$

where $g_1(\overline{0})$ and $g_2(0)$ are the eigenvalues. Then, by completing the square, we determine the coordinate transformation

$$\beta_1 = \tilde{g}_1(\xi_1,\xi_2)^{1/2}\,\xi_1 \pm \tilde{g}_1(\xi_1,\xi_2)^{-1/2}\,h(\xi_2)\xi_2^{\,2}$$
$$\beta_2 = [\tilde{g}_2(\xi_2) \mp \tilde{g}_1(\xi_1,\xi_2)^{-1}\,h(\xi_2)^2]^{1/2}\,\xi_2\ , \tag{5.2.3}$$

where $\tilde{g}_1(\overline{\xi}) = g_1(\overline{\xi})$ when $g_1(\overline{0}) > 0$, and $\tilde{g}_1(\overline{\xi}) = -g_1(\overline{\xi})$ when $g_1(\overline{0}) < 0$. The positive sign in β_1 occurs when the Hessian of $\phi(\overline{r},\overline{p})$ is positive definite at $(\overline{r}_o,\overline{p}_o)$. The positive sign in β_2 occurs when the Hessian of $\phi(\overline{r},\overline{p})$ is definite indefinite at $(\overline{r}_o,\overline{p}_o)$, with $g_2(0)$ the positive eigenvalue. Otherwise, the negative signs appear in the transformation. At $(\overline{r}_o,\overline{p}_o)$, Equation (5.2.3) carries $\phi(\overline{r},\overline{p})$ at $(\overline{r}_o,\overline{p}_o)$ to

$$\tilde{\phi}(\overline{r}_o,\beta_1,\beta_2) = \phi(\overline{r}_o,\overline{0}) \pm \beta_1^{\,2} \pm \beta_2^{\,2}\ , \tag{5.2.4}$$

where the sign of each β_i is determined by whether the corresponding eigenvalue is positive or negative.

5.3 Singular Stationary Points

If the Hessian determinant of $\phi(\overline{r},\overline{p})$ is zero at the stationary point, $\phi(x,y,p_x,p_y)$ at $(\overline{r}_o,\overline{p}_o)$ is transformed to

$$\tilde{\phi}(\overline{r}_o,\beta_1,\beta_2) = \phi(\overline{r}_o,\overline{0}) \pm \beta_1^{\,2} \pm \beta_2^{\,n}\ ,$$

where $n \geq 3$. If $n = 3$ or 4, the coordinate transformation carrying $\phi(\overline{r},\overline{p})$ at $(\overline{r}_o,\overline{p}_o)$ to $\tilde{\phi}(\overline{r}_o,\beta_1,\beta_2)$ may again be obtained by explicit algebraic computation. For $n > 4$, the determination of the appropriate coordinate transformation requires the algorithm specified in the proof of the Gromoll-Meyer Lemma, i.e., implicit rather than explicit solutions of equations, e.g., [49]. The procedure can be made explicit, however, by an application of the Cauchy inversion formula [36].

Let

$$\bar{e}_\lambda = \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix}, \qquad \bar{e}_0 = \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix}$$

be the eigenvectors corresponding to the eigenvalues at λ and 0 (zero), respectively. Then, the principal axis transformation in Equation (5.2.2) and Taylor's theorem carry $\phi(\bar{r},\bar{p})$ at $(\bar{r}_o,\bar{p}_o)$ to

$$\bar{\phi}(\bar{r}_o,\xi_1,\xi_2) = \phi(\bar{r}_o,\bar{0}) + g(\xi_1,\xi_2)\xi_1{}^2 + 2\xi_1\xi_2{}^2 h(\xi_2) + \ell(\xi_2)\xi_2{}^n, \quad (5.3.1)$$

where n equals 3 or 4, $g(\bar{0}) = \lambda$ and $\ell(0) \neq 0$. When $n = 3$, by completing the square, we determine the coordinate transformation

$$\begin{aligned} \beta_1 &= \tilde{g}(\xi_1,\xi_2)^{1/2}\xi_1 \pm \tilde{g}(\xi_1,\xi_2)^{-1/2} h(\xi_2)\xi_2{}^2 \\ \beta_2 &= [\ell(\xi_2) - g(\xi_1,\xi_2)^{-1} h(\xi_2)^2\xi_2]^{1/3}\xi_2 , \end{aligned} \quad (5.3.2)$$

where $\tilde{g}(\xi_1,\xi_2) = g(\xi_1,\xi_2)$ when $g(\bar{0}) > 0$, and $\tilde{g}(\xi_1,\xi_2) = -g(\xi_1,\xi_2)$ when $g(\bar{0}) < 0$. The sign in β_1 is determined by the sign of λ. At $(\bar{r}_o,\bar{p}_o)$, Equations (5.3.2) carry $\phi(\bar{r}_o,\bar{p}_o)$ to

$$\tilde{\phi}(\bar{r}_o,\beta_1,\beta_2) = \phi(\bar{r}_o,\bar{0}) \pm \beta_1{}^2 + \beta_2{}^3 , \quad (5.3.3)$$

where the sign of β_1 is determined by sign of λ. Similarly, when $n = 4$, we determine the coordinate transformations

$$\begin{aligned} \beta_1 &= \tilde{g}(\xi_1,\xi_2)^{1/2}\xi_1 \pm \tilde{g}(\xi_1,\xi_2)^{-1/2} h(\xi_2)\xi_2{}^2 \\ \beta_2 &= [\ell(\xi_2) - g(\xi_1,\xi_2)^{-1}h(\xi_2)^2]^{1/4}\xi_2 , \end{aligned} \quad (5.3.4)$$

for $\ell(0) - g(\bar{0})^{-1}h(0)^2 > 0$ and

$$\beta_1 = \tilde{g}(\xi_1,\xi_2)^{1/2}\xi_1 \pm \tilde{g}(\xi_1,\xi_2)^{-1/2}h(\xi_2)\xi_2^{\,2}$$

$$\beta_2 = -[g(\xi_1,\xi_2)^{-1}h(\xi_2)^2 - \ell(\xi_2)]^{1/4}\xi_2 \tag{5.3.5}$$

for $\ell(0) - g(\overline{0})^{-1}h(0)^2 < 0$, where $\tilde{g}(\xi_1,\xi_2) = g(\xi_1,\xi_2)$ when $g(\overline{0}) > 0$, and $\tilde{g}(\xi_1,\xi_2) = -g(\xi_1,\xi_2)$ when $g(\overline{0}) < 0$. The positive sign in β_1 corresponds to $\lambda > 0$. At $(\overline{r}_o,\overline{p}_o)$, Equations (5.3.4) or (5.3.5) carry $\phi(\overline{r}_o,\overline{p}_o)$ to

$$\tilde{\phi}(\overline{r}_o,\beta_1,\beta_2) = \phi(\overline{r}_o,\overline{0}) \pm \beta_1^{\,2} \pm \beta_2^{\,4}\ , \tag{5.3.6}$$

where the sign of β_1 is determined by sign of λ and the sign of β_2 is determined by the sign of $\ell(0) - g(\overline{0})^{-1}h(0)^2$.

When $n > 4$, explicit algebraic computation does not, in general, suffice to determine a coordinate transformation which carries $\phi(\overline{r},\overline{p})$ at $(\overline{r}_o,\overline{p}_o)$ to

$$\tilde{\phi}(\overline{r}_o,\beta_1,\beta_2) = \phi(\overline{r}_o,\overline{0}) \pm \beta_j^{\,2} \pm \beta_k^{\,n}\ .$$

The procedure for determining the transformation can be made explicit, however. Application of the principal axis transformation in Equation (5.2.2) to $\phi(\overline{r},\overline{p})$ at $(\overline{r}_o,\overline{p}_o)$ determines

$$\overline{\phi}(\overline{r}_o,\xi_1,\xi_2) = \phi(\overline{r}_o,\overline{0}) + g(\xi_1,\xi_2)\xi_1^{\,2} + 2\xi_1\xi_2^{\,2}h(\xi_1,\xi_2) + \ell(\xi_2)\xi_2^{\,n}, \tag{5.3.7}$$

where $g(\overline{0})$ is the non-vanishing eigenvalue and $\ell(0) \neq 0$. Because $\partial_1^2\phi(\overline{0}) \neq 0$, i.e., $g(\overline{0}) \neq 0$, $\partial_1^{\,2}\overline{\phi}(\overline{0}) \neq 0$; hence, from the Implicit Function Theorem [21], setting $\partial_1\overline{\phi}(\xi_1,\xi_2) = 0$ determines an implicit equation for ξ_1 in terms of ξ_2, i.e.,

$$\xi_1 = \Theta(\xi_2)\ .$$

$\Theta(\xi_2)$ may be determined explicitly, however, from the Cauchy inversion formula [35], i.e.,

$$\Theta(\xi_2) = \frac{1}{2\pi i} \int_\gamma \frac{z\partial_1{}^2\bar{\phi}(z,\xi_2)dz}{\partial_1\bar{\phi}(z,\xi_2)} , \tag{5.3.8}$$

for fixed ξ_2, where γ is a circle in $\xi_2 = 0$ enclosing $\xi_1 = 0$. Then introducing the transformation

$$\begin{aligned} \xi_1 &= \alpha_1 + \Theta(\alpha_2) \\ \xi_2 &= \alpha_2 , \end{aligned} \tag{5.3.9}$$

Equation (5.3.7) becomes

$$\begin{aligned} \bar{\phi}[\bar{r}_o,\alpha_1 + \Theta(\alpha_2),\alpha_1] &= \phi(\bar{r}_o,\bar{0}) + g[\alpha_1 + \Theta(\alpha_2),\alpha_2][\alpha_1 + \Theta(\alpha_2)]^2 + \\ &\quad 2[\alpha_1 + \Theta(\alpha_2)]\alpha_2{}^2 h[\alpha_1 + \Theta(\alpha_2),\alpha_2] + \ell(\alpha_2)\alpha_2{}^n . \end{aligned} \tag{5.3.10}$$

Because the Taylor series of an analytic function of two variables may be written as a Taylor series in one variable with the other fixed [35], expanding Equation (5.3.10) about $\alpha_1 = 0$ with α_2 fixed determines

$$\begin{aligned} \bar{\phi}[\bar{r}_o,\alpha_1 + \Theta(\alpha_2),\alpha_2] &= \phi(\bar{r}_o,\bar{0}) + \bar{\phi}[\bar{r}_o,\Theta(\alpha_2),\alpha_2] + \\ &\quad \alpha_1\partial_1\bar{\phi}[\bar{r}_o,\Theta(\alpha_2),\alpha_2] + \alpha_1{}^2R(\alpha_1,\alpha_2) , \end{aligned} \tag{5.3.11}$$

where $R(\alpha_1,\alpha_2)$ is the remainder of the Taylor series less a factor of $\alpha_1{}^2$, and $R(\bar{0}) \neq 0$ [since $\partial_1{}^2\phi(\bar{0}) \neq 0$]. Because $\partial_1\bar{\phi}(\xi_1,\xi_2) = 0$, Equation (5.3.11) becomes

$$\bar{\phi}[\bar{r}_o,\alpha_1 + \Theta(\alpha_2),\alpha_2] = \phi(\bar{r}_o,\bar{0}) + \bar{\phi}[\bar{r}_o,\Theta(\alpha_2),\alpha_2] + \alpha_1{}^2R(\alpha_1,\alpha_2) . \tag{5.3.12}$$

Then the transformation

$$\gamma_1 = \alpha_1 |R(\alpha_1,\alpha_2)|^{1/2}$$
$$\gamma_2 = \alpha_2 \tag{5.3.13}$$

obtains

$$\hat{\phi}(\bar{r}_o,\gamma_1,\gamma_2) = \phi(\bar{r}_o,\bar{0}) \pm \gamma_1{}^2 + \mu(\gamma_2) = \bar{\phi}[\bar{r}_o,\alpha_1, + \Theta(\alpha_2),\alpha_2] \ . \tag{5.3.14}$$

Finally, since the first non-vanishing Taylor coefficient (at $\xi_2 = 0$) in the ξ_2-direction is the n-th [Equation (5.3.7)], $\mu(\gamma_2)$ may be expressed as

$$\mu(\gamma_2) = \gamma_2{}^n \tilde{\mu}(\gamma_2) \ ,$$

where $\tilde{\mu}(\bar{0}) \neq 0$. Consequently, the transformation

$$\beta_1 = \gamma_1$$
$$\beta_2 = \gamma_2 |\tilde{\mu}(\gamma_2)|^{1/n} \tag{5.3.15}$$

carries $\tilde{\phi}(\bar{r}_o,\gamma_1,\gamma_2)$ to

$$\tilde{\phi}(\bar{r}_o,\beta_1,\beta_2) = \phi(\bar{r}_o,\bar{0}) \pm \beta_1{}^2 \pm \beta_2{}^n \ , \tag{5.3.16}$$

where the signs of the β_i must be determined for each specific case, as above. Equation (5.3.16) is the form of the phase required for the determination of the asymptotic series, c.f. Equation (5.1.2).

5.4 Transformation of the Amplitude

Under the coordinate transformation which carries $\phi(\bar{r},\bar{p})$ to the appropriate canonical form, the integral in Equation (5.1.1) at $(\bar{r}_o,\bar{p}_o)$ becomes

$$\iint A(x,y,p_x,p_y,\tau)\exp\{i\tau\phi(x,y,p_x,p_y)\}dp_x dp_y =$$

$$\exp\{i\tau\phi(\bar{r}_o,\bar{p}_o)\}\iint \tilde{A}(\bar{r}_o,\beta_1,\beta_2,\tau)\exp\{i\tau(\pm\beta_1{}^2 \pm \beta_2{}^n)\}d\beta_1 d\beta_2 \ , \tag{5.4.1}$$

where $\tilde{A}(r_o,\beta_1,\beta_2,\tau) = A(\bar{r}_o,p_x,p_y,\tau) \cdot [\frac{\partial(p_x,p_y)}{\partial(\beta_1,\beta_2)}]$, which is the form required for the determination of the asymptotic series. Since $A(\bar{r},\bar{p},\tau)$ is analytic over the region of integration, $\tilde{A}(\bar{r}_o,\beta_1,\beta_2,\tau)$ may be expressed as

$$\tilde{A}(\bar{r}_o,\beta_1,\beta_2,\tau) = \sum_{rs} b_{rs}\beta_1{}^r\beta_2{}^s \ , \tag{5.4.2}$$

with

$$b_{rs} = -\frac{1}{4\pi^2}\iint_{\delta\,\varepsilon}\frac{A(\bar{r}_o,p_x,p_y,\tau)dp_x dp_y}{\gamma_1(p_x,p_y)^{r+1}\gamma_2(p_x,p_y)^{s+1}} \ , \tag{5.4.3}$$

where

$$\begin{aligned}\gamma_1 &= \beta_i[\xi_1(p_x,p_y),\ \xi_2(p_x,p_y)] \\ \gamma_2 &= \beta_2[\xi_1(p_x,p_y),\ \xi_2(p_x,p_y)] \ ,\end{aligned} \tag{5.4.4}$$

and δ and ε are contours enclosing the origin. Equation (5.4.3), which determines the b_{rs} explicitly, proceeds from an extension of the Cauchy inversion formula to two variables.

More specifically, in one complex variable, if $f(z)$ is analytic and single-valued at z_o, and $f^{-1}(z)$ exists and is single-valued in a neighborhood of $w_o = f(z_o)$, the Cauchy inversion formula takes the form

$$g(f^{-1}(w)) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(z)f'(z)dz}{f(z)-w_o}, \tag{5.4.5}$$

where γ is a circle enclosing z_o. If $F(z_1,z_2)$ is an analytic function of two complex variables, it may be regarded as an analytic function of z_1 for fixed z_2 or as an analytic function of z_2 for fixed z_1 [35]. The Implicit Function Theorem implies that two analytic functions each of two complex variables, e.g.,

$$w_1 = g_1(z_1,z_2)$$

$$w_2 = g_2(z_1,z_2)$$

with

$$\frac{\partial(w_1,w_2)}{\partial(z_1,z_2)} (\bar{0}) \neq 0 ,$$

may be inverted to determine

$$\begin{aligned} z_1 &= \Theta_1(w_1,w_2) \\ z_2 &= \Theta_2(w_1,w_2) . \end{aligned} \tag{5.4.7}$$

By repeated application of the Cauchy inversion formula, however Equation (5.4.7) may be determined explicitly. Heuristically, first z_1 is determined as a function of z_2 and w_1, e.g., $z_1 = h(w_1,z_2)$. Next z_2 is determined as a function of w_1 and w_2. Finally, substituting for z_2 in $z_1 = h(w_1,z_2)$ determines z_1 as a function of w_1 and w_2.

More explicitly,

$$z_1 = h(w_1,z) = \frac{1}{2\pi i} \int_{c} \frac{\eta \partial_1 g_1(\eta,z)d\eta}{g_1(\eta,z)-w_1} , \tag{5.4.8}$$

where c is a contour enclosing the origin and, for clarity, it is assumed $g_1(\bar{0}) = g_2(\bar{0}) = 0$ and $\partial_1 g_1(\bar{0}) \neq 0$. Since $w_2 = g_2(z_1,z_2)$

$$w_2 = g_2[h(w_1,z_2),z_2] \; , \tag{5.4.9}$$

$$z_2 = \Theta_2(w_1,w_2) = \frac{1}{2\pi i}\int_{c'} \frac{\{\partial_1 g_2[h(w_1,\eta),\eta]\partial_2 h(w_1,\eta) + \partial_2 g_2[h(w_1,\eta),\eta]\}\eta d\eta}{g_2[h(w_1,\eta),\eta] - w_2} \; , \tag{5.4.10}$$

where c' is a contour enclosing the origin. Then

$$z_1 = \Theta(w_1,w_2) = h[w_1,\Theta_2(w_1,w_2)] \; . \tag{5.4.11}$$

An analytic function of two complex variables, $f(z_1,z_2)$, may be written directly as a function of two other complex variables, $F(w_1,w_2)$, i.e.,

$$f(z_1,z_2) = F(w_1,w_2) = f[\Theta_1(w_1,w_2),\Theta_2(w_1,w_2)] \; ,$$

by an analogous procedure. The Cauchy inversion formula implies that

$$F(w_1,w_2) = \frac{1}{2\pi i}\int_{c'} \frac{f[h(w_1,\eta),\eta]\{\partial_1 g_2[h(w_1,\eta),\eta]\partial_2 h(w_1,\eta) + \partial_2 g_2[h(w_1,\eta),\eta]\}d\eta}{g_2[h(w_1,\eta),\eta] - w_2} \; . \tag{5.4.12}$$

However,

$$\begin{aligned} &\partial_1 g_1[h(w_1,\eta),\eta] \cdot \partial_1 h(w_1,\eta) = 1 \\ &\partial_1 g_1[h(w_1,\eta),\eta] \cdot \partial_2 h(w_1,\eta) + \partial_2 g_1[h(w_1,\eta),\eta] = 0 \; . \end{aligned} \tag{5.4.13}$$

Substituting Equations (5.4.13) into Equation (5.4.12),

$$F(w_1,w_2) = \frac{1}{2\pi i}\int_{c'} \frac{f(h,\eta)\{-\partial_1 g_2 \partial_2 g_1 + \partial_1 g_1 \partial_2 g_2\} d\eta}{(\partial_1 g_1)(g_2 - w_2)} \quad , \tag{5.4.14}$$

where

$$h = h(w_1,\eta)$$

$$g_1 = g_1[h(w_1,\eta),\eta]$$

$$g_2 = g_2[h(w_1,\eta),\eta] \ .$$

But

$$-\partial_1 g_2[h(w_1,\eta),\eta]\partial_2 g_1[h(w_1,\eta),\eta] + \partial_1 g_1[h(w_1,\eta),\eta]\partial_2 g_2[h(w_1,\eta),\eta]$$

$$= J[h(w_1,\eta),\eta] \ ,$$

i.e., the Jacobian. Therefore,

$$F(w_1,w_2) = \frac{1}{2\pi i}\int_{c'} \frac{f[h(w_1,\eta),\eta]J[h(w_1,\eta),\eta]d\eta}{\partial_1 g_1[h(w_1,\eta),\eta](g_2[h(w_1,\eta),\eta]-w_2)} . \tag{5.4.15}$$

Now applying the Cauchy inversion formula to the integrand in Equation (5.4.15), i.e.,

$$\frac{f[h(w_1,\eta),\eta]J[h(w_1,\eta),\eta]}{\partial_1 g_1[h(w_1,\eta),\eta](g_2[h(w_1,\eta),\eta]-w_2)} =$$

$$\frac{1}{2\pi i}\int_{c} \frac{f(\xi,\eta)\dfrac{\partial(w_1,w_2)}{\partial(\xi,\eta)}\, d\xi d\eta}{[g_1(\xi,\eta)-w_1][g_2(\xi,\eta)-w_2]} \tag{5.4.16}$$

determines

$$F(w_1,w_2) = -\frac{1}{4\pi^2}\iint\limits_{c\ c'} \frac{f(\xi,\eta)\,\dfrac{\partial(w_1,w_2)}{\partial(\xi,\eta)}\,d\xi d\eta}{[g_1(\xi,\eta)-w_1][g_2(\xi,\eta)-w_2]} \ . \tag{5.4.17}$$

If $f(\xi,\eta) = \tilde{A}(\bar{r}_o,\beta_1,\beta_2,\tau)$, the amplitude, then

$$b_{rs} = -\frac{1}{4\pi^2}\iint\limits_{c\ c'} \frac{\tilde{A}(\bar{r}_o,\gamma_1(\xi,\eta),\gamma_2(\xi,\eta),\tau)\,\dfrac{\partial(\gamma_1,\gamma_2)}{\partial(\xi,\eta)}\,d\xi d\eta}{\gamma_1(\xi,\eta)^{r+1}\ \gamma_2(\xi,\eta)^{s+1}} \ . \tag{5.4.18}$$

Since

$$\tilde{A}(\bar{r}_o,\gamma_1(\xi,\eta),\ \gamma_2(\xi,\eta),\tau)\ \frac{\partial(\gamma_1,\gamma_2)}{\partial(\xi,\eta)} = A(\bar{r}_o,\xi,\eta,\tau) \ ,$$

Equation (5.4.18) is equivalent to Equation (5.4.3) with $c=\varepsilon$, $c'=\delta$, cf. [21].

CHAPTER VI

ASYMPTOTIC EXPANSIONS OF THE INTEGRALS

6.1 Introduction

Consider the integral determined in Chapter V,

$$I = \iint \tilde{A}(\bar{r}_o,\beta_1,\beta_2,\tau)\exp\{i\tau(\pm\beta_1{}^2 \pm \beta_2{}^n)\}d\beta_1 d\beta_2 \ . \tag{6.1.1}$$

When $n = 2$, the stationary phase technique applies directly. When $n \geq 3$, the classical technique must be modified to determine the asymptotic expansion. In this chapter, the asymptotic expansion of the above integral is determined for all n.

6.2 Duistermaat's Theorem

As was noted in Chapter I, there are several approaches to the classical stationary phase technique. A computationally simple approach derives from an elegant theorem of Duistermaat [22]. Specifically, let Q(a) be a symmetric, diagonal matrix with non-zero eigenvalues, and let $g(x,a,\tau)$ be a bounded function with bounded derivatives which is analytic near the stationary points of the phase. Then, for $\tau \to \infty$, Duistermaat's Theorem determines the asymptotic expansion

$$\iint g(\bar{x},a,\tau)\exp\{i\tau < Q(a)x,x/2>\}d\bar{x} \sim$$

$$\left(\frac{2\pi}{\tau}\right)^{n/2}|\det Q(a)|^{-1/2}\exp\{i(\pi/4)\operatorname{sgn}Q(a)\}\sum_{k=0}\frac{1}{k!}D^k g(\bar{0},a)\tau^{-k}, \tag{6.2.1}$$

where

$$D = \frac{i}{2} < |\det Q(a)|^{-1} \frac{\partial}{\partial x}, \frac{\partial}{\partial x} > ,$$

sgnQ(a) denotes the number of positive eigenvalues of Q(a) less the number of negative eigenvalues, and n is the dimension of the region of integration. Applying Equation (6.2.1) to the integral in Equation (6.1.1) determines

$$\iint \tilde{A}(\bar{r}_o,\beta_1,\beta_2,\tau)\exp\{i\tau(\pm\beta_1^2\pm\beta_2^2)\}d\beta_1 d\beta_2$$

$$\sim (2)^{1/2}\frac{\pi}{\tau}\exp\{\pm i(\pi/4)\}\sum_{k=0}\frac{1}{k!}D^k\tilde{A}(\bar{r}_o,\bar{0})\tau^{-k}$$

where

$$D = \frac{i}{2}\left(\pm\frac{\partial^2}{\partial\beta_1^2} \pm \frac{\partial^2}{\partial\beta_2^2}\right) ,$$

with the sign of $\frac{\partial}{\partial\beta i}$ corresponding to the sign of β_i, and it has been assumed that both β_1 and β_2 have the same sign (otherwise, sgnQ(a) = 0 and $\exp\{i\pi\,\mathrm{sgn}Q(a)/4\} = 1$). Duistermaat's Theorem determines the complete asymptotic series at any regular point.

6.3 The Modified Stationary Phase Technique

When $n \geq 3$, i.e., Q(a) has one zero eigenvalue, the theorem of Duistermaat cannot be applied directly. The asymptotic series of Equation (6.1.1) may yet be obtained by a modification of Duistermaat's approach. Let Equation (6.1.1) be expressed as

$$\iint \tilde{A}(\bar{r}_o,\beta_1,\beta_2,\tau)\exp\{i\tau(\pm\beta_1^2\pm\beta_2^n)\}d\beta_1 d\beta_2$$

$$= \int \exp\{\pm i\tau\beta_2^n\}d\beta_2 \int \tilde{A}(\bar{r}_o,\beta_1,\beta_2,\tau)\exp\{\pm i\tau\beta_1^2\}d\beta_1 . \qquad (6.3.1)$$

The asymptotic series of the integral over β_1 proceeds from Equation (6.2.1). Each series coefficient is a function of β_2; hence, each term determines an integral over β_2. The asymptotic series of each integral may be determined by a modification of the technique outlined in Chapter I. The complete asymptotic series is the sum of the series of the individual integrals over β_2.

More explicitly, applying Duistermaat's Theorem, in Equation (6.2.1), to the integral over β_1 in Equation (6.3.1) obtains

$$\int \tilde{A}(\bar{r}_o, \beta_1, \beta_2, \tau) \exp\{\pm i\tau\beta_1^{\,2}\} d\beta_1$$
$$\sim \left(\frac{\pi}{\tau}\right)^{1/2} \exp\{\pm i(\pi/4)\} \sum_{k=0} \left(\frac{1}{k!}\right)\left(\frac{i}{2}\right)^k \bar{A}_{2k}(\bar{r}_o, 0, \beta_2)\tau^{-k} , \tag{6.3.2}$$

where

$$\bar{A}_{2k}(\bar{r}_o, 0, \beta_2) = \frac{\partial^{2k}\tilde{A}(\bar{r}_o, 0, \beta_2, \tau)}{\partial\beta_1^{\,2k}}$$

Each $\bar{A}_{2k}(\bar{r}_o, 0, \beta_2)$ determines an integral of the form

$$\bar{I} = \int \bar{A}_{2k}(\bar{r}_o, 0, \beta_2) \exp\{\pm i\tau\beta_2^{\,n}\} d\beta_2 \ . \tag{6.3.3}$$

Since $A(\bar{r}, \beta_1, \beta_2, \tau)$ and all its derivatives are bounded (Chapter I), the integral in Equation (6.3.3) may be expressed as

$$I(\tau)\bar{A}_{2k}(\bar{r}_o, 0, \beta_2) = \int_{-\infty}^{\infty} \bar{A}_{2k}(\bar{r}_o, 0, \beta_2) \exp\{\pm i\tau\beta_2^{\,n}\} d\beta_2 \ . \tag{6.3.4}$$

Expanding $\bar{A}_{2k}(\bar{r}_o, 0, \beta_2)$ about $\beta_2 = 0$ determines

$$\int_{-\infty}^{\infty} \bar{A}_{2k}(\bar{r}_o,0,\beta_2)\exp\{\pm i\tau\beta_2{}^n\}d\beta_2 = \alpha_o\int_{-\infty}^{\infty}\exp\{\pm i\tau\beta_2{}^n\}d\beta_2 + \alpha_1\int_{-\infty}^{\infty}\exp\{\pm i\tau\beta_2{}^n\}\beta_2 d\beta_2 +$$

$$\ldots + \alpha_{n-2}\int_{-\infty}^{\infty}\exp\{\pm i\tau\beta_2{}^n\}\beta_2{}^{n-2}d\beta_2 + \frac{1}{n}\int_{-\infty}^{\infty}\exp\{\pm i\tau\beta_2{}^n\}\beta_2{}^{n-1}R\bar{A}_{2k}(\bar{r}_o,0,\beta_2)d\beta_2, \tag{6.3.5}$$

where

$$\alpha_j = \left(\frac{1}{j!}\right)\frac{d^j}{d\beta_2{}^j}\bar{A}_{2k}(\bar{r}_o,\bar{0})$$

and

$$R\bar{A}_{2k}(\bar{r}_o,0,\beta_2) = n\beta_2{}^{1-n}\{\bar{A}_{2k}(\bar{r}_o,0,\beta_2) - \bar{A}_{2k}(\bar{r}_o,\bar{0}) - \sum_{j=0}^{n-2}\beta_2{}^j\left(\frac{1}{j!}\right)\frac{d^j}{d\beta_2{}^j}\bar{A}_{2k}(\bar{r}_o,\bar{0})\},$$

i.e., the remainder of the Taylor series less a factor of n^{-1}.

The integrals

$$J_j(\tau) = \int_{-\infty}^{\infty}\exp\{\pm i\tau\beta_2{}^n\}\beta_2{}^j d\beta_2 \tag{6.3.6}$$

are determined by contour integration (Appendix B). A partial integration of

$$\frac{1}{n}\int_{-\infty}^{\infty}\exp\{\pm i\tau\beta_2{}^n\}\beta_2{}^{n-1}R\bar{A}_{2k}(\bar{r}_o,0,\beta_2)d\beta_2$$

obtains

$$\frac{1}{n}\int_{\infty}^{\infty}\exp\{\pm i\tau\beta_2{}^n\}\beta_2{}^{n-1}R\bar{A}_{2k}(\bar{r}_o,0,\beta_2)d\beta_2 = \frac{1}{i\tau}\int_{\infty}^{\infty}\left(\frac{d}{d\beta_2}\exp\{\pm i\tau\beta_2{}^n\}\right)R\bar{A}_{2k}(\bar{r}_o,0,\beta_2)d\beta_2 =$$

$$\pm \exp\{\pm i\tau\beta_2{}^n\} R\overline{A}_{2k}(\overline{r}_o,0,\beta_2)\Big|_{-\infty}^{\infty} \mp \frac{1}{i\tau}\int_{-\infty}^{\infty} \exp\{\pm i\tau\beta_2{}^n\}\frac{d}{d\beta_2}[R\overline{A}_{2k}(\overline{r}_o,0,\beta_2)]d\beta_2 =$$

$$\mp \frac{1}{i\tau}\int_{-\infty}^{\infty} \exp\{\pm i\tau\beta_2{}^n\}\frac{d}{d\beta_2}[R\overline{A}_{2k}(\overline{r}_o,0,\beta_2)]d\beta_2, \tag{6.3.7}$$

where the last equality follows from the boundedness of $A(\overline{r},\overline{p},\tau)$ and its derivatives [28]. Let S be an operator defined by

$$S\overline{A}_{2k}(\overline{r}_o,0,\beta_2) = \frac{d}{d\beta_2}\,[R\overline{A}_{2k}(\overline{r}_o,\overline{0},\beta_2)]\ . \tag{6.3.8}$$

Then, Equation (6.3.5) may be expressed as

$$I(\tau)\overline{A}_{2k}(\overline{r}_o,0,\beta_2) = \alpha_o J_o + \alpha_1, J_1 + \ldots + \alpha_{n-2}J_{n-2} \mp \frac{1}{i\tau} I(\tau)S\overline{A}_{2k}(\overline{r}_o,0,\beta_2)\ , \tag{6.3.9}$$

or, alternatively, as an operator equation

$$I(\tau)(\hat{1} \pm \frac{1}{i\tau}S) = \hat{\alpha}_o J_o(\tau) + \hat{\alpha}_1 J_1(\tau) + \ldots + \hat{\alpha}_{n-2}J_{n-2}(\tau)\ , \tag{6.3.10}$$

where $\hat{1}$ is the identity operator and the $\hat{\alpha}_k$ are the operators which carry functions to constants. Because Equation (6.3.10) is a power series, the operator $(\hat{1} \pm \frac{1}{i\tau}S)$ has a (right) inverse; namely $\sum_{\ell=0}^{\infty} (\mp\tau)^{-\ell}S^{\ell}$. Thus,

$$I(\tau) = (\sum_{k=0}^{\infty} \hat{\alpha}_k J_k(\tau))(\sum_{k=0}^{\infty} (\mp i\tau)^{-\ell}S^{\ell})\ . \tag{6.3.11}$$

Therefore, Equation (6.3.5) becomes

$$I(\tau)\bar{A}_{2k}(\bar{r}_o,0,\beta_2) = \int_{-\infty}^{\infty}\bar{A}_{2k}(\bar{r}_o,0,\beta_2)\exp\{\pm i\tau\beta_2{}^n\}d\beta_2 =$$

$$J_o(\tau)\sum_{\ell=0}^{\infty}(\mp i\tau)^{-\ell}S^{\ell}\bar{A}_{2k}(\bar{r}_o,0,\beta_2) + J_1(\tau)\sum_{\ell=0}^{\infty}(\mp i\tau)^{-\ell}S^{\ell}\bar{A}_{2k}(\bar{r}_o,0,\beta_2) + \ldots$$

$$+ J_{n-2}(\tau)\sum_{\ell=0}^{\infty}(\mp i\tau)^{-\ell}S^{\ell}\bar{A}_{2k}(\bar{r}_o,0,\beta_2) = (\sum_{j=0}^{n-2}J_j(\tau))[\sum_{\ell=0}^{\infty}(\mp i\tau)^{-\ell}S^{\ell}\bar{A}_{2k}(\bar{r}_o,0,\beta_2)]$$

$$= \sum_{j=0}^{n-2}J_j(\tau)[\sum_{\ell=0}^{\infty}(\mp i\tau)^{-\ell}(\frac{d}{d\beta_2}R)^{\ell}\bar{A}_{2k}(\bar{r}_o,0,\beta_2)] \ . \tag{6.3.12}$$

Thus, combining Equations (6.3.2) and (6.3.12) determines

$$\iint\tilde{A}(\bar{r}_o,\beta_1,\beta_2,\tau)\exp\{i\tau(\pm\beta_1{}^2\pm\beta_2{}^n)\}d\beta_1 d\beta_2 \sim$$

$$(\frac{\pi}{\tau})^{1/2}\exp\{i(\pi/4)\}\sum_{k=0}^{\infty}\frac{1}{k!}(\frac{i}{2})^k\tau^{-k}\{\sum_{j=0}^{n-2}J_j(\tau)[\sum_{\ell=0}^{\infty}(\mp i\tau)^{-\ell}(\frac{d}{d\beta_2}R)^{\ell}\bar{A}_{2k}(\bar{r}_o,\bar{0})]\} \ .$$

(6.3.13)

For $n \geq 3$, Equation (6.3.13) is the complete asymptotic series.

CHAPTER VII

CONCLUDING REMARKS

7.1 Introduction

In this chapter, the complete algorithm is summarized. An example with an explicit emitter geometry, boundary conditions on the momentum and a specific medium profile is presented. Finally, some promising extensions of this investigation are outlined.

7.2 Summary of the Algorithm

We consider wave propagation in a medium characterized by the reduced Helmholtz wave equation

$$\frac{\partial^2\psi(x,y)}{\partial x^2} + \frac{\partial^2\psi(x,y)}{\partial y^2} + \tau f(x)\psi(x,y) = 0 \ , \tag{1.3.1a}$$

where x is the depth, y is the range, and f(x) is the profile. We assume that

$$\psi(x,y) = \iint A(x,y,p_x,p_y,\tau)\exp\{i\tau\phi(x,y,p_x,p_y)\}dp_x dp_y \ , \tag{1.3.2a}$$

where

$$A(x,y,p_x,p_y,\tau) \sim \sum_k A_k(x,y,p_x,p_y)\,\tau^{-k} \ ,$$

and $\phi(x,y,p_x,p_y)$ has the form

$$\phi(x,y,p_x,p_y) = xp_x + yp_x - S(p_x,p_y) \ . \tag{3.3.7}$$

Let $y = g(x)$ specify the geometry of the emitter centered at $(\bar{x},\bar{y})$, with the boundary momentum, $\bar{p} = (\bar{p}_x,\bar{p}_y)$, and the spatial derivatives (in the x-direction) of the momentum at the emitter being known.

The algorithm begins by forming the Hamiltonian

$$H = p_x^{\,2} + p_y^{\,2} - f(x) = 0 \ . \tag{3.3.1}$$

Then the Lagrange manifold is determined from Equations (3.3.2), (3.3.3), and (3.3.4)

$$x = \frac{\partial S}{\partial p_x} = f^{-1}(p_x^{\,2} + p_y^{\,2}) \tag{7.2.1a}$$

$$y = \int \frac{\partial^2 S}{\partial p_x \partial p_y}\, dp_x = \frac{\partial \tilde{S}}{\partial p_y} + \Theta(p_y) = \frac{\partial S}{\partial p_y} \ , \tag{7.2.1b}$$

where $\Theta(p_y)$ is an arbitrary function which, for simplicity, we assume a polynomial in p_y with constant coefficients. These coefficients are determined by noting that on the emitter, Equations (7.2.1a) and (7.2.1b) become

$$\overline{x} = \frac{\partial S(\overline{p})}{\partial p_x} = f^{-1}(\overline{p}_x^{\,2} + \overline{p}_y^{\,2}) \tag{7.2.2a}$$

$$\overline{y} = \frac{\partial \tilde{S}(\overline{p})}{\partial p_y} + \Theta(\overline{p}_y) = \frac{\partial S(\overline{p})}{\partial p_y} \ . \tag{7.2.2b}$$

Solving Equation (7.2.2a) for p_x and substituting into Equation (7.2.2b) determines an equation in x and p_y, e.g.,

$$g(x) = G(x, p_y) \ , \tag{7.2.3}$$

where x and p_y are both known on the emitter. Successive differentiations of Equation (7.2.3) with respect to x, evaluated at the emitter, lead to a system of linear algebraic equations for the coefficients in $\Theta(p_y)$ in terms of $\overline{x}$, $\overline{p}_y$ and the spatial variations of the momentum on the emitter. Solving for the coefficients determines $S(p_x, p_y)$ and, hence, $\phi(x, y, p_x, p_y)$.

The equation of the caustic in momentum space is determined from the Hessian of $\phi(x,y,p_x,p_y)$, i.e.,

$$\det \begin{bmatrix} \frac{\partial^2\phi}{\partial p_x{}^2} & \frac{\partial^2\phi}{\partial p_x \partial p_y} \\ \frac{\partial^2\phi}{\partial p_y \partial p_x} & \frac{\partial^2\phi}{\partial p_y{}^2} \end{bmatrix} = 0 \ , \tag{7.2.4}$$

and is projected onto coordinate space through the Lagrange manifold, Equations (7.2.1a) and (7.2.1b). That is, any $(\tilde{p}_x,\tilde{p}_y)$ satisfying Equation (7.2.4) determines a corresponding point $(\tilde{x},\tilde{y})$ in coordinate space defined by

$$\tilde{x} = \frac{\partial S(\tilde{p})}{\partial p_x} \ , \quad \tilde{y} = \frac{\partial S(\tilde{p})}{\partial p_y} \ .$$

The locus of these points, $(\tilde{x},\tilde{y})$, defines the caustic in coordinate space.

The asymptotic analysis of the field, i.e., the integral in Equation (1.3.2a), at any point $(\bar{r}_o,\bar{p}_o)$ proceeds from an analysis of $\phi(\bar{r},\bar{p})$ at $(\bar{r}_o,\bar{p}_o)$, (Appendix A). The determination of the asymptotic series of the integral at $(\bar{r}_o,\bar{p}_o)$ requires that $\phi(\bar{r},\bar{p})$ at $(\bar{r}_o,\bar{p}_o)$ be transformed to its canonical form (Chapter V). The asymptotic series of the transformed integral

$$\tilde{I} = \iint \tilde{A}(\bar{r}_o,\beta_1,\beta_2,\tau)\exp\{i\tau(\pm\beta_1{}^2 \pm \beta_2{}^n)\}d\beta_1 d\beta_2 \ , \tag{7.2.5}$$

where

$$\tilde{A}(\bar{r}_o,\beta_1,\beta_2,\tau) \sim \sum_k \tilde{A}_k(\bar{r}_o,\beta_1,\beta_2)\tau^{-k} \ ,$$

with the Jacobian included implicitly, requires the determination of the asymptotic series for each of

$$\tilde{I}_k = \iint \tilde{A}_k(\bar{r}_o,\beta_1,\beta_2)\exp\{i\tau(\pm\beta_1{}^2\pm\beta_2{}^n)\}d\beta_1 d\beta_2. \quad (7.2.6)$$

The asymptotic series of each integral is determined in Chapter VI. The $\tilde{A}_k(\bar{r}_o,\beta_1,\beta_2)$ are determined from the transport equation (Chapter IV). The complete asymptotic series of integral in Equation (7.2.5) is the sum of the asymptotic series of each integral in Equation (7.2.6) for all $k \geq 0$.

7.3 Example of the Algorithm

Consider a medium represented by a linear profile, $f(x) = x$. We investigate the far field, i.e., the distances involved are much larger than the dimension of the emitter, taken here to be a line source centered at (5,-4) with slope $y = 2x$ producing a wave of constant amplitude, $A(\bar{r},\bar{p}) = 1$. Let the magnitude of the momentum at the emitter be $|p| = 5$; for definiteness, let the components of the momentum and the spatial variation of the momentum at (5,-4) be

$$\begin{array}{llll} p_{xo} = -2.0 & & p_{yo} = 1.0 & \\ p'_{xo} = -0.67 & & p'_{yo} = -0.83 & \\ p''_{xo} = 2.05 & & p''_{yo} = 2.96 & \\ p'''_{xo} = 10.2 & & p'''_{yo} = 31.9 & \end{array} \quad (7.3.1)$$

where the primes indicate derivatives with respect to x. We represent the field by

$$\psi(x,y) = \iint A(x,y,p_x,p_y,\tau)\exp\{i\tau\phi(x,y,p_x,p_y)\}dp_x dp_y, \quad (1.3.2a)$$

where

$$A(x,y,p_x,p_y,\tau) \sim \sum_k A_k(x,y,p_x,p_y)\tau^{-k} ,$$

i.e., integral in Equation (1.3.2a) is an asymptotic solution of the reduced Helmholtz wave equation [Equation (1.3.1a)].

In this case, the Hamiltonian [Equation (3.3.1)] becomes

$$H = p_x^2 + p_y^2 - x = 0 \tag{7.3.2}$$

which leads to the Lagrange manifold [Equations (3.3.3) and (3.3.4)],

$$x = \frac{\partial S}{\partial p_x} = p_x^2 + p_y^2 \tag{7.3.3a}$$

$$y = \frac{\partial S}{\partial p_y} = 2p_x p_y + \Theta(p_y). \tag{7.3.3b}$$

For definiteness, let $\Theta(p_y)$ be a polynomial in p_y to the third order, i.e.,

$$\Theta(p_y) = a_1 + a_2 p_y + a_3 p_y^2 + a_4 p_y^3 \ . \tag{7.3.4}$$

The a_i in Equation (7.3.4) may be determined from Equations (7.3.1), (7.3.3a, b) and the emitter geometry, $y = 2x$. The substitution of p_x and $y = 2x$ into Equation (7.3.3b) leads to

$$2x = 2p_y \sqrt{x - p_y^2} + a_1 + a_2 p_y + a_3 p_y^2 + a_4 p_y^3 \ . \tag{7.3.5}$$

Three differentiations with respect to x then obtain a system of linear equations which, with the momentum conditions at (5,-4) specified by Equation (7.3.1), determine $a_1 = 3$, $a_2 = -5$, $a_3 = 1$, $a_4 = 1$. Thus, from Equations (7.3.3) and (7.3.4),

$$x = p_x^2 + p_y^2$$

$$y = 2p_x p_y + 3 - 5p_y + p_y^2 + p_y^3 \ , \tag{7.3.6}$$

and

$$\phi(x,y,p_x,p_y) = xp_x + yp_y - \frac{p_x^3}{3} - p_x p_y^2 - 3p_y + \frac{5}{2}p_y^2 - \frac{p_y^3}{3} - \frac{p_y^4}{4}. \qquad (7.3.7)$$

The equation of the caustic is determined by equating the Hessian determinant of $\phi(x,y,p_x,p_y)$ to zero, i.e.,

$$\det\begin{bmatrix} 2p_x & 2p_y \\ 2p_y & 2p_x - 5 + 2p_y + 3p_y^2 \end{bmatrix} = 0 ,$$

yielding

$$4p_x^2 - 4p_y^2 - 10p_x + 4p_x p_y + 6p_x p_y^2 = 0 . \qquad (7.3.8)$$

Equation (7.3.8) is the equation of the caustic in momentum space. Any $\tilde{p} = (\tilde{p}_x,\tilde{p}_y)$ satisfying Equation (7.3.8) determines a corresponding point $(\tilde{x},\tilde{y})$ in coordinate space defined by the Lagrange manifold, Equation (7.3.6),

$$\tilde{x} = \tilde{p}_x^2 + \tilde{p}_y^2$$

$$\tilde{y} = 2\tilde{p}_x\tilde{p}_y + 3 - 5\tilde{p}_y + \tilde{p}_y^2 + \tilde{p}_y^3 .$$

The locus of these $(\tilde{x},\tilde{y})$ is the equation of the caustic in coordinate space, Figure 1.

The asymptotic expansion of

$$\psi(x,y) = \iint A(x,y,p_x,p_y,\tau)\exp\{i\tau\phi(x,y,p_x,p_y)\}dp_x dp_y \qquad (1.3.2a)$$

proceeds by transforming the phase to its canonical form (Chapter V). We illustrate the procedure by determining the first two terms in the asymptotic series at $(\tilde{x},\tilde{y}) = (2,2)$, which, following Appendix A, is the cusp point. Hence, at (2,2) the canonical form of $\phi(x,y,p_x,p_y)$ is

$$\tilde{\phi}(2,2,\beta_1,\beta_2) = \phi(2,2,\tilde{p}_x,\tilde{p}_y) \pm \beta_1^2 \pm \beta_2^4 = \phi(2,2,p_x,p_y).$$

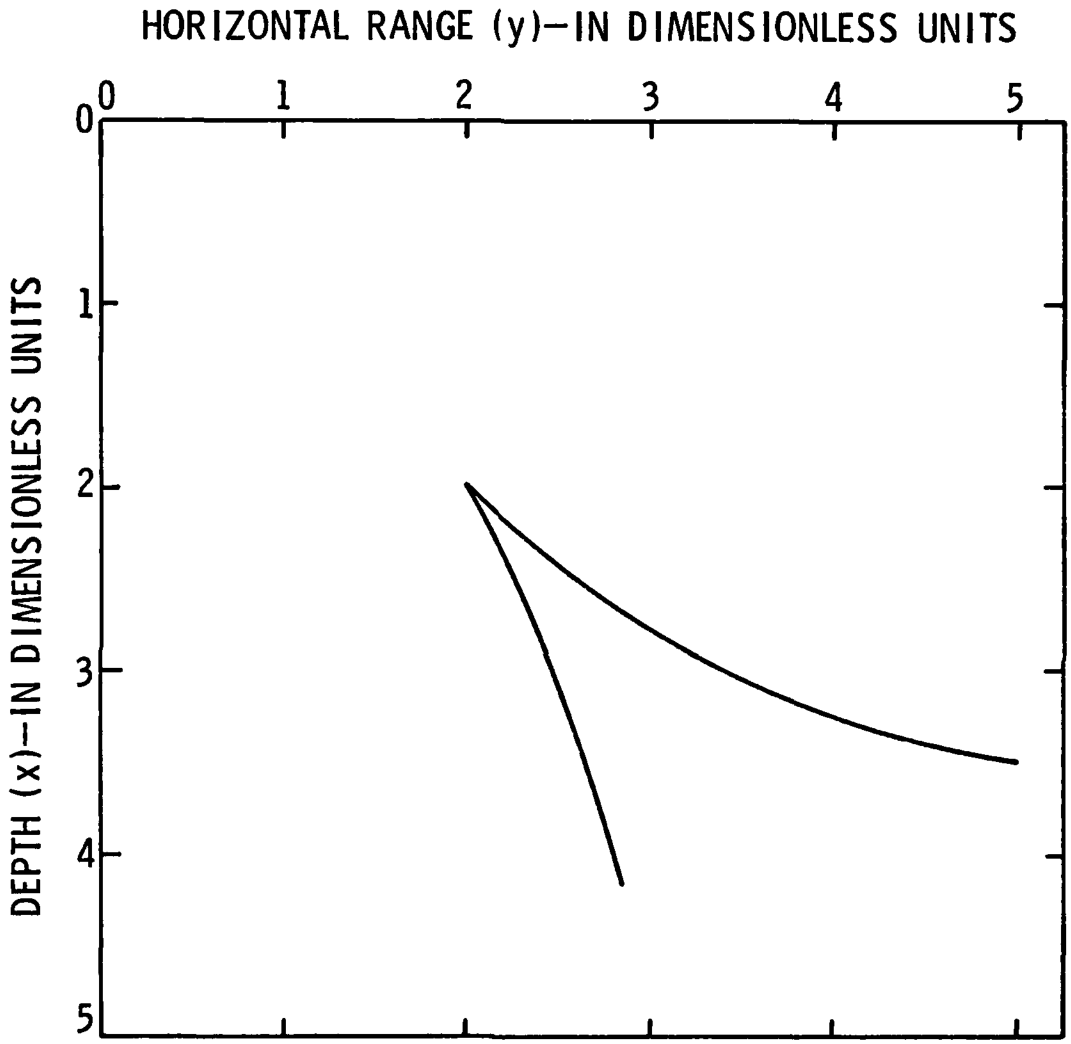


FIGURE 1. THE CAUSTIC CURVE

From the Lagrange manifold conditions, Equations (7.3.6), the corresponding momenta at $(\tilde{x},\tilde{y}) = (2,2)$ are $(p_x,p_y) = (1,1)$. Expanding $\phi(x,y,p_x,p_y)$ in a Taylor series at (2,2,1,1) yields

$$\phi(2,2,p_x,p_y) = \frac{19}{12} - (p_x-1)^2 - 2(p_x-1)(p_y-1) - (p_y-1)^2$$
$$-\frac{1}{3}(p_x-1)^3 - (p_x-1)(p_y-1)^2 - \frac{4}{3}(p_y-1)^3 - \frac{1}{4}(p_y-1)^4 .$$

Translating the origin to $(p_x,p_y) = (1,1)$ and applying the principal axis transformation, Equation (5.2.2), obtains

$$\overline{\phi}(2,2,\alpha_1,\alpha_2) = \frac{19}{12} - \alpha_2{}^2(1 - \frac{5}{8}\alpha_1 + \frac{1}{3}\alpha_2 + \frac{3}{32}\alpha_1{}^2 - \frac{1}{16}\alpha_1\alpha_2 + \frac{1}{64}\alpha_2{}^2)$$
$$-2\alpha_2(\frac{1}{32}\alpha_1 - \frac{1}{4})\alpha_1 - \frac{1}{64}\alpha_1{}^4.$$

Then, the coordinate transformation

$$\begin{aligned} \beta_1 &= \alpha_2(1 - \frac{5}{8}\alpha_1 + \frac{1}{3}\alpha_2 + \frac{3}{32}\alpha_1{}^2 - \frac{1}{16}\alpha_1\alpha_2 + \frac{1}{64}\alpha_2{}^2)^{1/2} \\ &\quad -\alpha_1{}^2(\frac{1}{32}\alpha_1 - \frac{1}{4})(1 - \frac{5}{8}\alpha_1 + \frac{1}{3}\alpha_2 + \frac{3}{32}\alpha_1{}^2 - \frac{1}{16}\alpha_1\alpha_2 + \frac{1}{64}\alpha_2{}^2)^{-1/2} \\ \beta_2 &= \alpha_1\{(\frac{1}{32}\alpha_1 - \frac{1}{4})^2(1 - \frac{5}{8}\alpha_1 + \frac{1}{3}\alpha_2 + \frac{3}{32}\alpha_1{}^2 - \frac{1}{16}\alpha_1\alpha_2 + \frac{1}{64}\alpha_2{}^2)^{-1} - \frac{1}{64}\}^{1/4} \end{aligned} \tag{7.3.9}$$

carries $\phi(\overline{r},\overline{p})$ at (2,2,1,1) to

$$\tilde{\phi}(2,2,\beta_1,\beta_2) = \frac{19}{12} - \beta_1{}^2 + \beta_2{}^4 .$$

Under the coordinate transformation in Equation (7.3.9), the integral in Equation (1.3.2a) becomes

$$\psi(2,2) = \iint \tilde{A}(2,2,\beta_1,\beta_2,\tau)\exp\{i\tau(\frac{19}{12} - \beta_1{}^2 + \beta_2{}^4)\}d\beta_1 d\beta_2 , \tag{7.3.10}$$

where

$$\tilde{A}(2,2,\beta_1,\beta_2,\tau) \sim \sum_k \tilde{A}_k(2,2,\beta_1,\beta_2)\tau^{-k} ,$$

with the Jacobian $\frac{\partial(\alpha_1,\alpha_2)}{\partial(\beta_1,\beta_2)}$, included implicitly, e.g., Chapter V.

The determination of the asymptotic series proceeds from Equations (6.3.2) and (6.3.5) and Appendix B, with $n = 4$. The first two terms of the asymptotic series at $(x,y) = (2,2)$ are then

$$\psi(2,2) \sim -0.77\exp\{\frac{11\pi i}{6}\}\Gamma(\frac{1}{4})\tau^{-3/4}\{\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\}$$
$$-0.54\exp\{\frac{11\pi i}{6}\}\Gamma(\frac{3}{4})\tau^{-5/4}\{\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\}.$$

7.4 Suggestions for Future Research

In this investigation, a partial differential equation of classical physics, the reduced Helmholtz wave equation, has been studied using theorems from topology, more specifically, singularity theory, in an essential way. Singularity theory is a well-developed research area in abstract mathematics, e.g., [25]; since the announcement of Thom's Classification Theorem, singularity theory has been the subject of intense interest among pure and applied mathematicians and scientists [52]. In the course of this investigation, a number of promising extensions of this research have become apparent.

The mathematics of this investigation focused on the reduced Helmholtz wave equation in two spatial variables. The analysis led to caustics which could be characterized by the A-series Thom potentials. An obvious generalization is to consider the reduced Helmholtz wave equation in three variables, which would lead to caustics corresponding

to the D-series (umbilic) Thom potentials. The determination of the caustic geometry in coordinate space would proceed as in Chapter III except that the phase, $\phi(\bar{r},\bar{p})$ would require a Hessian of rank 3. The field would be represented by an integral of the form

$$\psi(x,y,z) = \iiint A(x,y,z,p_x,p_y,p_z)\exp\{i\tau\phi(x,y,z,p_x,p_y,p_z)\}dp_x dp_y dp_z, \tag{7.4.1}$$

where

$$A(x,y,z,p_x,p_y,p_z\tau) \sim \sum_k A_k(x,y,z,p_x,p_y,p_z)\tau^{-k}.$$

At those field points where the Hessian of $\phi(\bar{r},\bar{p})$ was completely degenerate, the integral in Equation (7.4.1) must be transformed to an integral whose phase was the appropriate umbilic (Appendix A). The stationary phase technique developed above is not valid at such field points. Consequently, a further modification of the technique would be required to determine the asymptotic series.

The general stationary phase technique for multiple integrals could itself be studied. As was noted in Chapter I, integral in Equation (1.2.9) may be transformed to an integral of the form

$$\int \exp\{i\tau u\}dg(u) ,$$

where $g(u)$ is the integral of $A(\bar{r},\bar{p},\tau)$ over the region $\phi(\bar{r},\bar{p}) \le u$. The asymptotic behavior of the original integral is now determined by an analysis of the singular points of $g(u)$. It would be interesting to investigate the utility of Thom's Classification Theorem in the analysis of such integrals.

The physical phenomenology considered in this investigation was concerned with the propagation of unattenuated waves through a non-dispersive medium. An extension of the above algorithm to include attenuated waves propagating in a dispersive medium seems possible. Indeed, the extension of the algorithm to include other linear "wave equations", e.g., the Klein-Gordon equation, appears very promising.

As this investigation was concerned with classical physics, "semi-classical" wave phenomena, e.g., barrier penetration, were not considered. It appears, however, that the physical phenomenology motivating Maslov and Arnold was primarily semi-classical [2,4]. It would be interesting to determine if the above algorithm could be extended to include semi-classical wave mechanics, generalizing the results of Eckmann and Seneor [24].

BIBLIOGRAPHY

1. Abramowitz, M. and Stegun, I. A., Handbood of Mathematical Functions, Dover Publications, Inc., New York, (1965).

2. Arnold, V. I., "Characteristic Class Entering in Quantization Conditions," Functional Anal. Appl., 1, 1-13, (1967).

3. Arnold, V. I., "Singularities of Smooth Maps," Russian Math. Surveys, 23, 1-43, (1968).

4. Arnold, V. I., "Integrals of Rapidly Oscillating Functions and Singularities of Projections of Lagrangian Manifolds," Functional Anal. Appl., 6, 222-224, (1972).

5. Arnold, V. I., "Normal Forms for Functions Near Degenerate Critical Points, the Weyl Groups of A_n, D_k and E_k and Lagrangian Singularities," Functional Anal. Appl., 6, 254-272, (1972).

6. Arnold, V. I., "Remarks on the Stationary Phase Method and Coxeter Numbers," Russian Math. Surveys, 28, 19-48, (1973).

7. Arnold, V. I., "Normal Forms for Functions in the Neighborhood of Degenerate Critical Points," Russian Math. Surveys, 29, 10-50, (1974).

8. Arnold, V. I., "Critical Points of Smooth Functions," Proc. Internat. Congr. Math., 1, 19-39, Canadian Math. Congr., Vancouver, (1974).

9. Arnold, V. I., "Critical Points of Smooth Functions and Their Normal Forms," Russian Math. Surveys, 30, 1-75, (1975).

10. Arnold, V. I., Mathematical Methods of Classical Mechanics, trans. K. Vogtemann and A. Weinstein, Springer-Verlag, Berlin, (1978).

11. Bender, C. M. and Orszag, S. A., Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill Book Company, New York, (1978).

12. Berry, M. V., "Waves and Thom's Theorem," Adv. Phys. 25, 1-25, (1975).

13. Bieberbach, L., Differentialgluchungen (Differential Equations), Springer-Verlag, Berlin, (1930).

14. Birkhoff, G. D., "Quantum Mechanics and Asymptotic Series," Bull. Am. Math. Soc., 39, 681-700, (1933).

15. Bleistein, N. and Handlesman, R. A., Asymptotic Expansions of Integrals, Holt, Rinehart and Winston, New York, (1975).

16. Born, M. and Wolf, E., Principles of Optics, The Macmillan Company, New York, (1959).

17. Brekhovskii, L. M., Waves in Layered Media, Academic Press, New York, (1960).

18. Brocker, Th. and Lander, L., Differential Germs and Catastrophes, London Mathematical Society Lecture Notes 17, Cambridge University Press, Cambridge, (1975).

19. Chako, N., "Asymptotic Expansions of Double and Multiple Integrals Arising in Diffraction Theory," J. Inst. Math. Applics., 1, 371-405, (1965).

20. Corben, H. C. and Stehle, P., Classical Mechanics, John Wiley, New York, (1950).

21. Dieudonne, J., Foundations of Modern Analysis, Academic Press, New York, (1960).

22. Duistermaat, J. J., Fourier Integral Operators, Courant Institute Lecture Notes, New York University, New York, (1973).

23. Duistermaat, J. J., "Oscillatory Integrals, Lagrange Immersions and Unfolding of Singularities," Comm. Pure Appl. Math, 27, 207-281, (1974).

24. Eckmann, J. P. and Seneor, R., "The Maslov-WKB Method for the (an-) Harmonic Oscillator," Arch. Rat. Mech. Anal., 2, 153-173, (1973).

25. Eells, J., Singularities of Smooth Maps, Gordon and Breach, New York, (1967).

26. Goldstein, H., Classical Mechanics, Addison-Wesley, Reading, Massachusetts, (1950).

27. Gromoll, D. and Meyer, W., Topology, 8, 361-370, (1969).

28. Guillemin, V. and Sternberg, S., Geometric Asymptotics, American Mathematical Society, Providence, Rhode Island, (1977).

29. Haken, H., Synergetics, Springer-Verlag, Berlin, (1977).

30. Heading, J., Introduction to Phase-Integral Methods, Methuen and Co., Ltd., London, (1962).

31. Janich, K., "Caustics and Catastrophes," Math. Ann., 209, 161-180, (1974).

32. Jones, D. S. and Klein, M., "Asymptotic Expansions of Multiple Integrals and the Method of Stationary Phase," J. Math. Phys., 37, 1-28, (1958).

33. Keller, J. B., "A Geometrical Theory of Diffraction, Calculus of Variations and Its Applications," Proc. Symposia Appl. Math., 8, 27-54, McGraw-Hill Book Co., New York, (1958).

34. Ludwig, D., "Uniform Asymptotic Expansions at a Caustic," Comm. Pure Appl. Math., 19, 215-250, (1966).

35. Markushevich, A. I., Theory of Functions of a Complex Variable, trans. R. A. Silverman, Chelsea Publ. Co., New York, (1965).

36. Mather, J., "Stability of C^∞-Mappings I. The Division Theorem," Ann. Math., 87, 89-104, (1968).

37. Mather, J., "Stability of C^∞-Mappings II. Infinitesimal Stability Implies Stability," Ann. Math., 89, 254-291, (1969).

38. Mather, J., "Stability of C^∞-Mappings III. Finitely Determined Map Germs," Publ. Math. IHES, 35, 127-156, (1968).

39. Mather, J., "Stability of C^∞-Mappings IV. Classification of Stable Germs by R-Algebras," Publ. Math. IHES, 37, 223-248, (1969).

40. Mather, J., "Stability of C^∞-Mappings V. Transversality," Adv. Math., 4, 301-336, (1970).

41. Mather, J., "Stability of C^∞-Mappings VI. The Nice Dimensions," Proceedings of the Liverpool Singularities Symposium, Lecture Notes in Mathematics 192, ed. C.T.C. Wall, Springer-Verlag, Berlin, (1971).

42. Miller, F. H., Partial Differential Equations, John Wiley and Sons, New York, (1947).

43. Minorsky, N., Nonlinear Mechanics, J. W. Edwards, Ann Arbor, Michigan, (1947).

44. Morse, M., "The Critical Points of a Function of n Variables," Trans. Am. Math. Soc., 33, 72-91, (1933).

45. Morse, M. and Cairns, S. S., Critical Point Theory in Global Analysis and Differential Topology, Academic Press, New York, (1969).

46. Nayfeh, A. H., Perturbation Methods, John Wiley, New York, (1973).

47. Poincare, H., Les Method Nouvells de la Mechanique Celeste (New Methods of Celestial Mechanics), Gauthier-Villars, Paris, (1892).

48. Poincare, H., Sur Les Courbes Defines par une Equation Differentielle (On Curves Defines by a Differential Equation), Gauthier-Villars, Paris, (1928).

49. Poston, T. and Steward, I., Catastrophe Theory and Its Applications, Pitman, London, (1978).

50. Thom, R., "Topological Methods in Biology," Topology, 8, 313-335, (1969).

51. Thom, R., Structural Stability and Morphogenesis, trans. D. Fowler, Benjamin-Addison Wesley, New York (1975).

52. von Kampen, N. G., "Higher Corrections to the WKB Approximation," Physica, 35, 70-79, (1967).

53. Wasserman, G., Stability of Unfoldings, Lecture Notes in Mathematics 393, Springer-Verlag, Berlin, (1974).

54. Zauderer, E., "Uniform Asymptotic Solutions of the Reduced Wave Equation," J. Math. Anal. Appl., 30, 157-171, (1970).

APPENDIX A

THOM'S THEOREM AND THE CLASSIFICATION ALGORITHM

A.1 Introduction

Thom's Classification Theorem states that a class of functions $\phi(x,y)$, noted below, may be transformed into certain canonical forms, commonly referred to as the "Thom potentials". The equivalency between $\phi(x,y)$ and its corresponding Thom potential derives from topological considerations; specifically, one is obtained from the other through a diffeomorphism, a differentiable coordinate transformation with a differentiable inverse. $\phi(x,y)$, then, is merely in a different algebraic form than its corresponding Thom potential. In this appendix a formal statement of Thom's Theorem is given and an algorithm that determines the canonical form corresponding to a given function is detailed.

A.2 Thom's Classification Theorem

Up to the addition of a non-degenerate quadratic form in other variables and up to multiplication by ± 1, an r-parameter family of smooth functions, $1 \leq r \leq 4$, is equivalent up to a diffeomorphism to one of the following:

r	Degenerate Term	Universal Unfolding	Name
1	x^3	$x^3 + k_1x$	Fold
2	x^4	$x^4 + k_1x^2 + k_2x$	Cusp
3	x^5	$x^5 + k_1x^3 + k_2x^2 + k_3x$	Swallowtail
3	$x^3 - xy^2$	$x^3 - xy^2 + k_1y^2 + k_2x + k_3y$	Elliptic Umbilic
3	$x^3 + y^3$	$x^3 + y^3 + k_1xy + k_2x + k_3y$	Hyberbolic Umbilic
4	x^6	$x^6 + k_1x^4 + k_2x^3 + k_3x^2 + k_4x$	Butterfly
4	$x^2y + y^4$	$x^2y + y^4 + k_1x^2 + k_2y^2 + k_3x + k_4y$	Parabolic Umbilic

Thom's Theorem has been rigorously proven in a series of papers by Mather [36-41].

A.3 The Classification Algorithm

Consider the function $\phi(x,y)$ at $\bar{r}_o = (x_o, y_o)$, where

$$\nabla\phi(\bar{r}_o) = \frac{\partial\phi(r_o)}{\partial x}\hat{i} + \frac{\partial\phi(\bar{r}_o)}{\partial y}\hat{j} = 0$$

At $\bar{r}_o$ the Hessian of $\phi(x,y)$ becomes

$$d^2\phi(\bar{r}_o) = \begin{bmatrix} \dfrac{\partial^2\phi(\bar{r}_o)}{\partial x^2} & \dfrac{\partial^2\phi(\bar{r}_o)}{\partial x\partial y} \\ \dfrac{\partial^2\phi(\bar{r}_o)}{\partial y\partial x} & \dfrac{\partial^2\phi(\bar{r}_o)}{\partial y^2} \end{bmatrix}$$

CASE I: $\det(d^2\phi(\bar{r}_o)) \neq 0$.

$\bar{r}_o = (x_o, y_o)$ is a regular point. *Then there exists a coordinate* transformation in a neighborhood of $\bar{r}_o$, $(x,y) \to (\xi_1, \xi_2)$, such that

$$\tilde{\phi}(\xi_1, \xi_2) = \phi(\bar{r}_o) + \sum_{i=1}^{2} (\underline{+}1)\xi_i^2 ,$$

where the signs of ξ_i are determined by the eigenvalues of $d^2\phi(\bar{r}_o)$. The canonical form is that of $\tilde{\phi}(\xi_1, \xi_2)$, [44].

CASE II: $\det(d^2\phi(\bar{r}_o)) = 0$, $d^2\phi(\bar{r}_o) \neq 0$, i.e., at least one of $\partial_{ij}\phi(\bar{r}_o) \neq 0$, $i,j, = 1,2$.

Thom's Theorem allows this case to be described as if $\phi(x,y)$ were a function of a single variable. First, the zero eigenvector is determined from

$$d^2\phi(\bar{r}_o)\binom{e_1}{e_2} = 0 ,$$

i.e.,

$$\frac{\partial^2\phi(\bar{r}_o)}{\partial x^2} e_1 + \frac{\partial^2\phi(\bar{r}_o)}{\partial x\partial y} e_2 = 0$$

$$\frac{\partial^2\phi(\bar{r}_o)}{\partial x\partial y} e_1 + \frac{\partial^2\phi(\bar{r}_o)}{\partial y^2} e_2 = 0$$

If $e_1 = 1$, then

$$e_2 = -\frac{\dfrac{\partial^2\phi(\bar{r}_o)}{\partial x^2}}{\dfrac{\partial^2\phi(\bar{r}_o)}{\partial x\partial y}} e_1$$

Then form

$$f(t) = \phi(x_o + te_1,\ y_o + te_2).$$

The Thom potential corresponding to $\phi(x,y)$ at $\bar{r}_o$ is determined from the Taylor series of $f(t)$ at $t = 0$; specifically, the first non-vanishing term in the Taylor series determines the equivalent Thom potential. In a neighborhood of $\bar{r}_o$ there exists a coordinate transformation that carries $\phi(x,y)$ to the degenerate term of its corresponding Thom potential.

(i) If $f^{I}(0) = f^{II}(0) = 0$, $f^{III}(0) \neq 0$, the corresponding Thom potential is the fold.

(ii) If $f^{I}(0) = f^{II}(0) = f^{III}(0) = 0$, $f^{IV}(0) \neq 0$, the corresponding Thom potential is the cusp.

(iii) If $f^{I}(0) = f^{II}(0) = f^{III}(0) = f^{IV}(0) = 0$, $f^{V}(0) \neq 0$, the corresponding Thom potential is the swallowtail.

(iv) If $f^{I}(0) = f^{II}(0) = f^{III}(0) = f^{IV}(0) = f^{V}(0) = 0$, $f^{VI}(0) \neq 0$, the corresponding Thom potential is the butterfly.

(v) If $f^{I}(0) = f^{II}(0) = f^{III}(0) = f^{IV}(0) = f^{V}(0) = f^{VI}(0) = 0$, there is no corresponding Thom potential.

(It should be noted the case $f^{I}(0) = 0$, $f^{II}(0) \neq 0$ cannot occur here because $\det(d^2\phi(\bar{r}_o)) \neq 0$.)

CASE III: $d^2\phi(\bar{r}_o) = 0$, i.e., each $\partial_{ij}\phi(\bar{r}_o) = 0$, $i,j = 1,2$.

(i) If not only the second derivatives of $\phi(x,y)$ at $\bar{r}_o$ equal zero, but also the third derivatives, i.e., the third order terms in the Taylor series at $\bar{r}_o$, there is no corresponding Thom potential.

(ii) If the third derivatives of $\phi(x,y)$ at $\bar{r}_o$ are not all equal to zero, the third order Taylor series may be expressed as

$$a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 \ , \tag{A.3.1}$$

where the a_i are the appropriate Taylor series coefficients evaluated at $\bar{r}_o$. If $a_1 \neq 0$, dividing Equation (A.3.1) by y^3 and equating to zero determines

$$F(t) = a_1t^3 + a_2t^2 + a_3t + a_4 = 0 \ ,$$

where $t = x/y$. (If $a_1 = 0$ and $a_4 \neq 0$, interchanging x and y yields an analogous cubic.) Four root combinations lead to corresponding Thom potentials. That is, for four root combinations there exists a coordinate transformation in a neighborhood of $\bar{r}_o$ that carries $\phi(\bar{r})$ at $\bar{r}_o$ to the degenerate term on the Thom potential:

(a)	three real equal roots	:	fold, x^3
(b)	three real unequal roots	:	elliptic umbilic, $x^3 - xy^2$
(c)	three real roots, two equal	:	parabolic umbilic, $x^2y + y^4$
(d)	one real root and one complex conjugate pair	:	hyperbolic umbilic, $x^3 + y^3$

If $a_1 = a_4 = 0$, $a_2, a_3 \neq 0$ the corresponding Thom potential is the

parabolic umbilic. If $a_1 = a_4 = 0$ and one of a_2 or a_3 equal zero, there is no corresponding Thom potential.

APPENDIX B

DETERMINATION OF THE CONTOUR INTEGRALS

B.1 Introduction

In Chapter VI the determination of the asymptotic series required the evaluation of integrals of the form

$$I = \int_{-\infty}^{\infty} \exp\{i\tau x^n\} x^m dx \ , \tag{B.1.1}$$

where m and n are integers, $n > m+1$. These integrals are evaluated here.

B.2 Transformation to Common Form

Consider the integral in Equation (B.1.1). If m is odd and n is even, then, by symmetry

$$\int_{-\infty}^{\infty} \exp\{i\tau x^n\} x^m dx = 0 \ .$$

If m and n are both even,

$$\int_{-\infty}^{\infty} \exp\{i\tau x^n\} x^m dx = 2\int_{0}^{\infty} \exp\{i\tau x^n\} x^m dx \ . \tag{B.2.1}$$

Then by a change in variable to $t = \tau x^n$, Equation (B.2.1) becomes

$$2\int_{0}^{\infty} \exp\{i\tau x^n\} x^m dx = \frac{2}{n\tau^k}\int_{0}^{\infty} \frac{\exp\{it\}}{t^{1-k}}\, dt = \frac{2}{n\tau^k}\{\int_{0}^{\infty} \frac{\cos t dt}{t^{1-k}} + i\int_{0}^{\infty} \frac{\sin t dt}{t^{1-k}}\}, \tag{B.2.2}$$

where $k = (m+1)/n$.

If n is odd, express Equation (B.2.1) as

$$\int_{-\infty}^{\infty} \exp\{i\tau x^n\} x^m dx = \int_{-\infty}^{0} \exp\{i\tau x^n\} x^m dx + \int_{0}^{\infty} \exp\{i\tau x^n\} x^m dx. \tag{B.2.3}$$

Then, by a change of variables to $t = -\tau x^n$,

$$\int_{-\infty}^{0} \exp\{i\tau x^n\} x^m dx = \frac{(-1)^m}{n\tau^k}\{\int_0^\infty \frac{\cos t dt}{t^{1-k}} - i\int_0^\infty \frac{\sin t dt}{t^{1-k}}\}. \tag{B.2.4}$$

Similarly, by a change of variables to $t = \tau x^n$

$$\int_0^\infty \exp\{i\tau x^n\} x^m dx = \frac{1}{n\tau^k}\{\int_0^\infty \frac{\cos t dt}{t^{1-k}} + i\int_0^\infty \frac{\sin t dt}{t^{1-k}}\} . \tag{B.2.5}$$

B.3 Evaluation of the Integrals

The trigonometric integrals in Equations (B.2.2), (B.2.4) and (B.2.5) can be evaluated by considering the integral

$$I = \int_c \frac{\exp\{iz\}dz}{z^{1-k}} , \tag{B.3.1}$$

where c is the contour in Figure 2.

From Cauchy's Integral Theorem [35]

$$\int_c \frac{\exp\{iz\}dz}{z^{1-k}} = \int_\rho^R \frac{\exp\{ix\}dx}{x^{1-k}} + \int_{c_R} \frac{\exp\{iz\}dz}{z^{1-k}} + \int_R^\rho \frac{\exp\{-y\}idy}{(iy)^{1-k}} + \int_{c_\rho} \frac{\exp\{iz\}dz}{z^{1-k}} = 0 . \tag{B.3.2}$$

Consider first

$$\int_{c_R} \frac{\exp\{iz\}dz}{z^s}$$

where $s = 1 - k$. Since $z = R\exp\{i\Theta\}$, and hence $dz = iR\exp\{i\Theta\}d\Theta$,

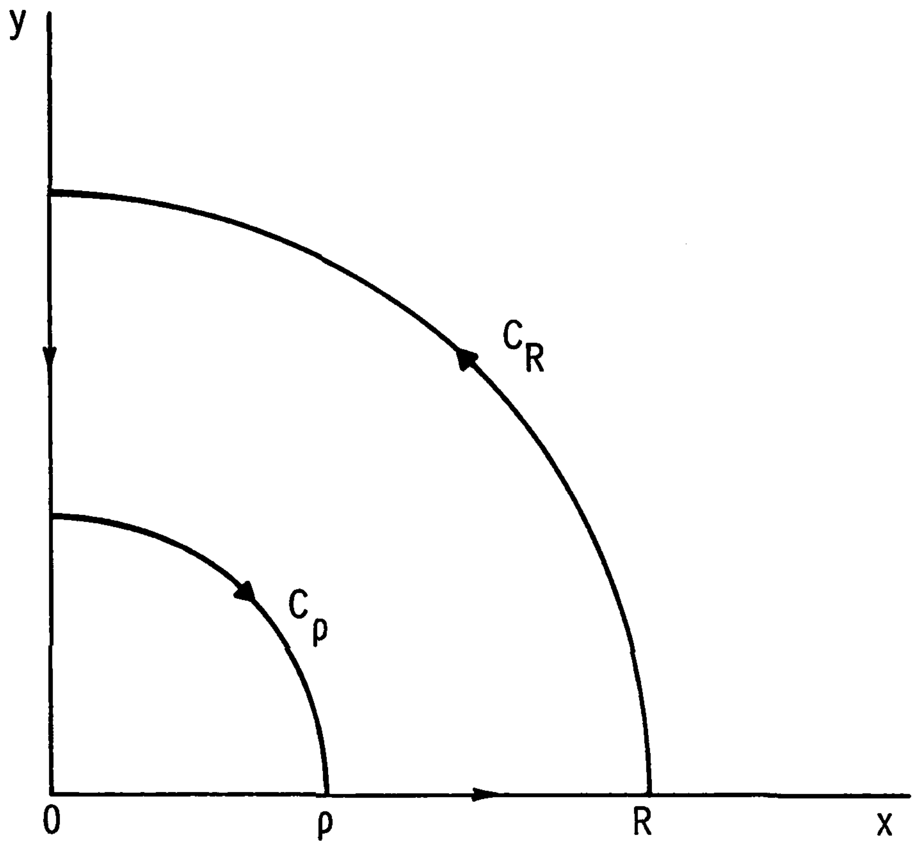


FIGURE 2. THE CONTOUR OF INTEGRATION

$$\int_{c_R} \frac{\exp\{iz\}dz}{z^s} = \int_{c_R} \frac{\exp\{iR\exp(i\Theta)\}iR\exp\{i\Theta\}d\Theta}{R^s\exp\{is\Theta\}}$$

$$= iR^{1-s}\int_{c_R} \exp\{iR(\cos\Theta+i\sin\Theta)\}[\cos k\Theta+i\sin k\Theta]d\Theta$$

$$\leq \left|iR^{1-s}\int_{c_R} \exp\{iR(\cos\Theta+i\sin\Theta)\}[\cos k\Theta+i\sin k\Theta]d\Theta\right| .$$

But

$$\left|iR^{1-s}\int_{c_R}\exp\{iR(\cos\Theta+i\sin\Theta)\}[\cos k\Theta+i\sin k\Theta]d\Theta\right| \leq R^{1-s}\int_{c_R}\exp\{-R\sin\Theta\}d\Theta$$

$$\leq R^{1-s}\int_0^{\pi/2}\exp\{-2R\Theta/\pi\}d\Theta ,$$

where the last inequality follows from $0 \leq \Theta \leq \pi/2$, hence $\sin\Theta \leq \pi/2$.

$$R^{1-s}\int_0^{\pi/2}\exp\{-2R\Theta/\pi\}d\Theta = \frac{\pi}{2}R^{-s}(1-\exp\{-R\}) = \frac{\pi}{2}R^{\frac{m+1-n}{n}}(1-\exp\{-R\}).$$

Now taking the limit as $R\to\infty$

$$\lim_{R\to\infty}\frac{\pi}{2}R^{\frac{m+1-n}{n}}(1-\exp\{-R\}) \to 0 ,$$

$$\therefore \quad \int_{c_R}\frac{\exp\{iz\}}{z^s}dz = 0 .$$

Next consider

$$\int_{c_\rho}\frac{\exp\{iz\}dz}{z^s} .$$

Since $z = \rho\exp\{i\Theta\}$, and hence $dz = i\rho\exp\{i\Theta\}d\Theta$,

$$\int_{c_\rho} \frac{\exp\{iz\}}{z^s}dz = \int_{c_\rho} \frac{\exp\{i\rho\exp(i\Theta)\}i\rho\exp\{i\Theta\}d\Theta}{\rho^s\exp\{is\Theta\}}$$

$$= \int_{c_\rho} \exp\{i\rho\exp(i\Theta)\}i\rho^{1-s}\exp\{i(1-s)\Theta\}d\Theta \ .$$

This last integral has no poles, therefore

$$|\exp\{i\rho\exp(i\Theta)\}i\rho^{1-s}\exp\{i(1-s)\Theta\}| < M\rho^{1-s}$$

where M is a constant. Because the length of the path around c_ρ is $\alpha\rho$, where α is the subtended angle,

$$|\exp\{i\rho\exp(i\Theta)\}i\rho^{1-s}\exp\{i(1-s)\Theta\}|d\Theta$$

$$= |\exp\{i\rho\exp(i\Theta)\}i\rho^{1-s}\exp\{i(1-s)\Theta\}|\alpha\rho < M\rho^{2-s} \ .$$

Now taking the limit as $\rho \to 0$,

$$\lim_{\rho\to 0} \alpha M\rho^{2-s} \to 0$$

$$\int_{c_\rho} \frac{\exp\{iz\}}{z^s}\, dz = 0 \ .$$

Therefore, from Equation (B.3.2)

$$\int^R \frac{\exp\{ix\}}{x^{1-k}}dx = -i\int_\rho^R \frac{\exp\{-y\}dy}{(iy)^{1-k}}$$

$$=[\cos(k\pi/2) + i\sin(k\pi/2)]\int_\rho^R \frac{\exp\{-y\}dy}{y^{1-k}} \ . \qquad \text{(B.3.3)}$$

But

$$\int_0^\infty \frac{\exp\{-y\}dy}{y^{1-k}} = \Gamma(k),$$

i.e., the Gamma Function, [1]; hence, taking the limit of Equation (B.3.3) as $\rho \to 0$ and $R \to \infty$, simultaneously, i.e.,

$$\lim_{\substack{R\to\infty \\ \rho\to 0}}\left\{\int_\rho^R \frac{\exp\{ix\}dx}{x^{1-k}} = [\cos(k\pi/2) + \sin(k\pi/2)]\int_\rho^R \frac{\exp\{-y\}dy}{y^{1-k}}\right\},$$

determines

$$\begin{aligned}
\int_0^\infty \frac{\cos x dx}{x^{1-k}} &= \Gamma(k)\cos(k\pi/2) = \Gamma\left(\frac{m+1}{n}\right)\cos[(m+1)\pi/2n] \\
\int_0^\infty \frac{\sin x dx}{x^{1-k}} &= \Gamma(k)\sin(k\pi/2) = \Gamma\left(\frac{m+1}{n}\right)\sin[(m+1)\pi/2n] .
\end{aligned} \tag{B.3.4}$$

Summarizing from Equations (B.2.2), (B.2.4), (B.2.5) and (B.3.4), if n is even

$$\int_{-\infty}^{\infty} \exp\{i\tau x^n\}x^m dx = \begin{cases} \dfrac{2\Gamma\left(\frac{m+1}{n}\right)}{n\tau^{\frac{m+1}{n}}}\{\cos[(m+1)\pi/2n] + i\sin[(m+1)\pi/2n]\} & m \text{ even} \\[2ex] 0 & m \text{ odd} \end{cases},$$

and if n is odd

$$\int_{-\infty}^{\infty} \exp\{i\tau x^n\}x^m dx = \begin{cases} \dfrac{2\Gamma\left(\frac{m+1}{n}\right)}{n\tau^{\frac{m+1}{n}}}\cos[(m+1)\pi/2n] & m \text{ even} \\[2ex] \dfrac{2\Gamma\left(\frac{m+1}{n}\right)}{n\tau^{\frac{m+1}{n}}}\sin[(m+1)\pi/2n] & m \text{ odd} \end{cases}.$$

DISTRIBUTION

Commander (NSEA 09G32) Naval Sea Systems Command Department of the Navy Washington, DC 20362	Copies 1 and 2
Commander (NSEA 0342) Naval Sea Systems Command Department of the Navy Washington, DC 20362	Copies 3 and 4
Defense Technical Information Center 5010 Duke Street Cameron Station Alexandria, VA 22314	Copies 5 through 16